

SINGULARITIES OF AXISYMMETRIC FREE SURFACE FLOWS WITH GRAVITY

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ABSTRACT. We consider a steady axisymmetric solution of the Euler equations for a fluid (incompressible and with zero vorticity) with a free surface, acted on only by gravity. We analyze stagnation points as well as points on the axis of symmetry. At points on the axis of symmetry which are not stagnation points, *constant velocity motion* is the only blow-up profile consistent with the invariant scaling of the equation. This suggests the presence of downward pointing cusps at those points.

At *stagnation points* on the axis of symmetry, the unique blow-up profile consistent with the invariant scaling of the equation is *Garabedian's pointed bubble solution* with water above air. Thus at stagnation points on the axis of symmetry with no water above the stagnation point, the invariant scaling of the equation cannot be the right scaling. A fine analysis of the blow-up velocity yields that in the case that the surface is described by an injective curve, the velocity scales almost like $\sqrt{X^2 + Y^2 + Z^2}$ and is asymptotically given by the velocity field

$$V(\sqrt{X^2 + Y^2}, Z) = c(-\sqrt{X^2 + Y^2}, 2Z)$$

with a nonzero constant c .

The last result relies on a frequency formula in combination with a concentration compactness result for the axially symmetric Euler equations by J.-M. Delort. While the concentration compactness result alone does *not* lead to strong convergence in general, we prove the convergence to be strong in our application.

1. INTRODUCTION

Consider the steady axisymmetric Euler equations for a fluid (incompressible and with zero vorticity) with a free surface acted on only by gravity. Using cylindrical coordinates and the Stokes stream function ψ (see for example [9, Exercise 4.18 (ii)]), we obtain the free boundary problem

$$\begin{aligned} \operatorname{div} \left(\frac{1}{x_1} \nabla \psi(x_1, x_2) \right) &= 0 \text{ in the water phase } \{\psi > 0\} \\ \frac{1}{x_1^2} |\nabla \psi(x_1, x_2)|^2 &= -x_2 \text{ on the free surface } \partial\{\psi > 0\}; \end{aligned} \tag{1.1}$$

here the original velocity field

$$V(X, Y, Z) = \left(-\frac{1}{x_1} \partial_2 \psi \cos \vartheta, -\frac{1}{x_1} \partial_2 \psi \sin \vartheta, \frac{1}{x_1} \partial_1 \psi \right),$$

where $(X, Y, Z) = (x_1 \cos \vartheta, x_1 \sin \vartheta, x_2)$.

Observe that the positive sign of ψ is chosen just for convenience and that replacing ψ by $-\psi$ our analysis covers the case of negative ψ as well.

Note also that the equations above describe apart from a model, where the fluid is pumped in or sucked out at a fixed boundary, also the case of a traveling wave traveling in the direction of the axis of symmetry; here the equations describe the steady flow in the moving frame, so that the original velocity field is

$$V(X, Y, Z, t) = \tilde{V}(X, Y, Z - c_0 t) + (0, 0, c_0),$$

where c_0 is the speed of the traveling wave and

$$\tilde{V}(X, Y, Z) = \left(-\frac{1}{x_1} \partial_2 \psi \cos \vartheta, -\frac{1}{x_1} \partial_2 \psi \sin \vartheta, \frac{1}{x_1} \partial_1 \psi \right).$$

[17] and [19], [18] are excellent reviews on two-dimensional water waves.

The free boundary problem (1.1) has been studied in [2] where regularity away from the degenerate sets $\{x_1 = 0\}$ (the axis of symmetry) and $\{x_2 = 0\}$ (containing all stagnation points) has been shown for minimizers of a certain energy.

In the present paper we will focus on precisely those two sets and analyze the profile of the velocity vector field close to points in those sets.

Due to the degeneracy of the free boundary condition $|\nabla \psi(x_1, x_2)|^2 = x_1^2 x_2$ at points $x^0 = (x_1^0, x_2^0)$ with $x_1^0 x_2^0 = 0$, we obtain *four* invariant scalings

$$\begin{aligned} & \frac{\psi(x^0 + rx)}{r} \text{ in the case } x_1^0 \neq 0 \text{ and } x_2^0 \neq 0, \\ & \frac{\psi(x^0 + rx)}{r^{\frac{3}{2}}} \text{ in the case } x_1^0 \neq 0 \text{ and } x_2^0 = 0, \\ & \frac{\psi(x^0 + rx)}{r^2} \text{ in the case } x_1^0 = 0 \text{ and } x_2^0 \neq 0, \\ & \frac{\psi(x^0 + rx)}{r^{\frac{5}{2}}} \text{ in the case } x_1^0 = x_2^0 = 0. \end{aligned}$$

Note that the velocity (in the moving frame) would scale like $1, |x|^{\frac{1}{2}}, 1, |x|^{\frac{1}{2}}$ in the respective cases.

In a first main result we determine the profile of the scaled solution as $r \rightarrow 0$ (Proposition 3.10): In the case $x_1^0 \neq 0$ and $x_2^0 \neq 0$ the only asymptotics possible is constant velocity flow parallel to the free surface. In the case $x_1^0 \neq 0$ and $x_2^0 = 0$ the only asymptotics possible is the well-known Stokes corner flow (see [4], [15], [16], [21]). Due to the perturbed equation the situation is actually not unlike the two-dimensional problem in the presence of vorticity (see [20], [5], [6], [7] for two-dimensional results in the presence of vorticity). In the case $x_1^0 = 0$ and $x_2^0 \neq 0$ the

only asymptotics possible is constant velocity flow in the gravity direction. This suggests the *possibility of air cusps* pointing in the gravity direction (Figure 1).

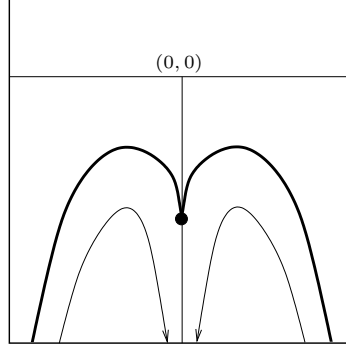


FIGURE 1. Dynamics suggested by our analysis

In the case $x_1^0 = x_2^0 = 0$ the only asymptotics possible is the *Garabedian pointed bubble solution* with water above air (cf. [10], Figure 2). This comes at first as a surprise as it means that there is no nontrivial asymptotic profile at all with air above water and with the invariant scaling. However there remains at this stage the possibility that the solution has a higher growth than that suggested by the invariant scaling.

In Theorem 3.12 we first analyze the possible shapes of the surface close to stagnation points and close to points on the axis of symmetry. Assuming that the surface is given by an injective curve and assuming also a strict Bernstein inequality (corresponding to a Rayleigh-Taylor condition) we obtain the following result:

In the case $x_1^0 \neq 0$ and $x_2^0 = 0$ the only asymptotics possible are the well-known Stokes corner (an angle of opening 120° in the direction of the axis of symmetry), and a horizontal point.

In the case $x_1^0 = 0$ and $x_2^0 < 0$ the only asymptotics possible are cusps in the direction of the axis of symmetry.

In the case $x_1^0 = x_2^0 = 0$ the only asymptotics possible are the *Garabedian pointed bubble asymptotics* (an angle of opening $\approx 114.799^\circ$ with water above air), and a *horizontal point*.

A fine analysis of the velocity profile in the last case ($x_1^0 = x_2^0 = 0$ and a horizontal point) is no mean feat, and we confine ourselves to the case of air above water. Here we prove (Theorem 7.1) that the velocity scales almost like $\sqrt{X^2 + Y^2 + Z^2}$ and is

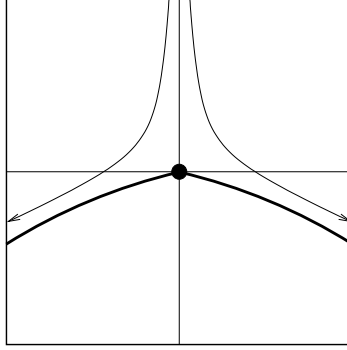


FIGURE 2. Garabedian pointed bubble asymptotics

asymptotically given by the velocity field

$$V(\sqrt{X^2 + Y^2}, Z) = c(-\sqrt{X^2 + Y^2}, 2Z),$$

where c is a nonzero constant (Figure 3).

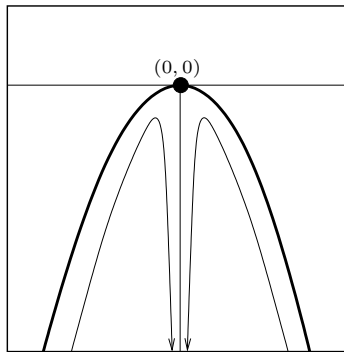


FIGURE 3. Dynamics suggested by our analysis

The proofs rely on a monotonicity formula as well as a *frequency formula* for the axisymmetric problem; as remarked in [21], it is for certain semilinear problems possible to derive *on the set of highest density* not a perturbation of Almgren's frequency formula (see [1], [13], [12], [11]), but a true nonlinear frequency formula. Here we extend the formula of [21] to the axisymmetric case. In combination with a concentration compactness result for the axially symmetric Euler equations by J.-M. Delort [8], this leads to the already mentioned profile for the velocity vector field. Note that while the concentration compactness result alone does *not* lead to strong convergence in general, we prove the convergence to the limiting velocity vector field to be strong in our application.

2. NOTATION

We will use coordinates (X, Y, Z) in the physical space \mathbf{R}^3 together with partial derivatives $\partial_X, \partial_Y, \partial_Z$ as well as two-dimensional coordinates $x = (x_1, x_2)$ together with partial derivatives ∂_1, ∂_2 . Sometimes we are going to use cylindrical coordinates $(X, Y, Z) = (x_1 \cos \vartheta, x_1 \sin \vartheta, x_2)$. We denote by $x \cdot y$ the Euclidean inner product in $\mathbf{R}^n \times \mathbf{R}^n$, by $|x|$ the Euclidean norm in \mathbf{R}^n , by $B_r(x^0) := \{x \in \mathbf{R}^n : |x - x^0| < r\}$ the ball of center x^0 and radius r , by $B_r^+(x^0) := \{x \in \mathbf{R}^n : x_1 > 0 \text{ and } |x - x^0| < r\}$, by $\partial B_r^+(x^0) := \{x \in \mathbf{R}^n : x_1 > 0 \text{ and } |x - x^0| = r\}$ and $\mathbf{R}_+^n := \{(x_1, \dots, x_n) : x_1 > 0\}$ the positive parts. *Note that $\partial B_r^+(x^0)$ is not the topological boundary of $B_r^+(x^0)$ and that $B_r^+(x^0)$ is not necessarily a half ball.*

We will use the notation B_r for $B_r(0)$ as well as B_r^+ for $B_r^+(0)$, and denote by ω_2 the 2-dimensional volume of B_1 .

We will use the weighted L^p space

$$L_w^p(\mathbf{R}_+^2) := \{v \text{ measurable} : \int_{\mathbf{R}_+^2} \frac{1}{x_1} |v|^p dx < +\infty\}$$

with norm $\|f\|_{L_w^p(\mathbf{R}_+^2)} = \left(\int_{\mathbf{R}_+^2} \frac{1}{x_1} |v|^p dx \right)^{\frac{1}{p}}$, the weighted Sobolev space

$$W_w^{1,p}(\mathbf{R}_+^2) := \{v \in L_w^p(\mathbf{R}_+^2) : \text{all weak partial derivatives of } v \\ \text{are contained in } L_w^p(\mathbf{R}_+^2)\}$$

as well as the local spaces

$$L_{w,loc}^p(\mathbf{R}_+^2) := \{v \text{ measurable} : \int_{B_R^+} \frac{1}{x_1} |v|^p dx < +\infty \text{ for each } R \in (0, +\infty)\}$$

and

$$W_{w,loc}^{1,p}(\mathbf{R}_+^2) := \{v \text{ measurable} : v \in L_{w,loc}^p(\mathbf{R}_+^2) \text{ and all weak partial derivatives} \\ \text{of } v \text{ are contained in } L_w^p(B_R^+) \text{ for each } R \in (0, +\infty)\}.$$

We denote by χ_A the characteristic function of a set A . For any real number a , the notation a^+ stands for $\max(a, 0)$ and a^- stands for $\min(a, 0)$. Also, \mathcal{L}^n shall

denote the n -dimensional Lebesgue measure and \mathcal{H}^s the s -dimensional Hausdorff measure. By ν we will always refer to the outer normal on a given surface. We will use functions of bounded variation $BV(U)$, i.e. functions $f \in L^1(U)$ for which the distributional derivative is a vector-valued Radon measure. Here $|\nabla f|$ denotes the total variation measure. Note that for a smooth open set $E \subset \mathbf{R}^2$, $|\nabla \chi_E|$ coincides with the surface measure on ∂E . We will also use the reduced boundary $\partial_{red} E$.

3. NOTION OF SOLUTION AND MONOTONICITY FORMULA

Let Ω be a bounded domain contained in $\{(x_1, x_2) : x_1 \geq 0\}$, in which to consider the combined problem for fluid and air. We study solutions u , in a sense to be specified, of the problem

$$\begin{aligned} \operatorname{div} \left(\frac{1}{x_1} \nabla u \right) &= \frac{\partial}{\partial x_1} \left(\frac{1}{x_1} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{1}{x_1} \frac{\partial u}{\partial x_2} \right) = 0 \quad \text{in } \Omega \cap \{u > 0\}, \\ \frac{1}{x_1^2} |\nabla u|^2 &= x_2 \quad \text{on } \Omega \cap \partial\{u > 0\}. \end{aligned} \quad (3.1)$$

Note that, compared to the Introduction, we have switched notation from ψ to u , and we have “reflected” the problem at the hyperplane $\{x_2 = 0\}$. Since our results are completely local, we do not specify boundary conditions on $\partial\Omega$.

We begin by introducing our notion of a *variational solution* of problem (3.1).

Definition 3.1 (Variational Solution). We define $u \in W_{w, \text{loc}}^{1,2}(\Omega)$ to be a *variational solution* of (3.1) if $u \in C^0(\Omega) \cap C^2(\Omega \cap \{u > 0\})$, $\nabla u / x_1 \in C^1(\Omega \cap \{u > 0\})$, $u = 0$ on $\{x_1 = 0\}$ (motivated by the fact that the velocity on the axis orthogonal to the axis direction should be zero), $u \geq 0$ in Ω , and the first variation with respect to domain variations of the functional

$$J(v) := \int_{\Omega} \left(\frac{1}{x_1} |\nabla v|^2 + x_1 x_2 \chi_{\{v > 0\}} \right) dx$$

vanishes at $v = u$, i.e.

$$\begin{aligned} 0 &= -\frac{d}{d\epsilon} J(u(x + \epsilon \phi(x)))|_{\epsilon=0} \\ &= \int_{\Omega} \left[\left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \operatorname{div} \phi - 2 \frac{1}{x_1} \nabla u D\phi \nabla u \right. \\ &\quad \left. + \left(-\frac{1}{x_1^2} |\nabla u|^2 + x_2 \chi_{\{u > 0\}} \right) \phi_1 + x_1 \chi_{\{u > 0\}} \phi_2 \right] dx \end{aligned}$$

for any $\phi = (\phi_1, \phi_2) \in C_0^1(\Omega; \mathbf{R}^2)$ such that $\phi_1 = 0$ on $\{x_1 = 0\}$.

A proof of the just mentioned first variation formula can be found in [14, Section 3.2]. An integration by parts shows that u satisfies on smooth parts of free boundary $\partial\{u > 0\}$ in $\{x_1 x_2 \neq 0\}$ the free boundary condition

$$\frac{1}{x_1} |\nabla u|^2 = x_1 x_2.$$

Theorem 3.2 (Monotonicity Formula). *Let u be a variational solution of (3.1), let $x^0 \in \Omega$ and let $\delta := \text{dist}(x^0, \partial\Omega)/2$. Let, for any $r \in (0, \delta)$,*

$$I(r) = \int_{B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) dx, \quad (3.2)$$

$$J(r) = \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u^2 d\mathcal{H}^1, \quad (3.3)$$

$$M^{int}(r) = r^{-2} I(r) - r^{-3} J(r), \quad (3.4)$$

$$M^{x_2}(r) = r^{-3} I(r) - \frac{3}{2} r^{-4} J(r), \quad (3.5)$$

$$M^{x_1}(r) = r^{-3} I(r) - 2r^{-4} J(r), \quad (3.6)$$

$$M^{x_1 x_2}(r) = r^{-4} I(r) - \frac{5}{2} r^{-5} J(r). \quad (3.7)$$

Then, for a.e. $r \in (0, \delta)$,

$$\begin{aligned} (M^{int}(r))' &= 2r^{-2} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{u}{r} \right)^2 d\mathcal{H}^1 \\ &\quad + r^{-3} \int_{B_r^+(x^0)} \left(-\frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + [(x_1 - x_1^0)x_2 + (x_2 - x_2^0)x_1] \chi_{\{u>0\}} \right) dx \\ &\quad + r^{-4} \int_{\partial B_r^+(x^0)} \frac{x_1 - x_1^0}{(x_1)^2} u^2 d\mathcal{H}^1. \end{aligned} \quad (3.8)$$

In the case $x_2^0 = 0$,

$$\begin{aligned} (M^{x_2}(r))' &= 2r^{-3} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{3}{2} \frac{u}{r} \right)^2 d\mathcal{H}^1 \\ &\quad + r^{-4} \int_{B_r^+(x^0)} \left(-\frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + (x_1 - x_1^0)x_2 \chi_{\{u>0\}} \right) dx \\ &\quad + \frac{3}{2} r^{-5} \int_{\partial B_r^+(x^0)} \frac{x_1 - x_1^0}{(x_1)^2} u^2 d\mathcal{H}^1. \end{aligned} \quad (3.9)$$

In the case $x_1^0 = 0$,

$$\begin{aligned} (M^{x_1}(r))' &= 2r^{-3} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - 2 \frac{u}{r} \right)^2 d\mathcal{H}^1 \\ &\quad + r^{-4} \int_{B_r^+(x^0)} (x_2 - x_2^0)x_1 \chi_{\{u>0\}} dx. \end{aligned} \quad (3.10)$$

Last, in the case $x_1^0 = x_2^0 = 0$,

$$(M^{x_1 x_2}(r))' = 2r^{-4} \int_{\partial B_r^+(0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{5}{2} \frac{u}{r} \right)^2 d\mathcal{H}^1. \quad (3.11)$$

Remark 3.3. (i) The integrand in the first integral on the right-hand side of (3.8) is a scalar multiple of $(\nabla u(x) \cdot (x - x^0) - u(x))^2$, and therefore vanishes if and only if u is a homogeneous function of degree 1 with respect to x^0 .

- (ii) The integrand in the first integral on the right-hand side of (3.9) is a scalar multiple of $(\nabla u(x) \cdot (x - x^0) - \frac{3}{2}u(x))^2$, and therefore vanishes if and only if u is a homogeneous function of degree $3/2$ with respect to x^0 .
- (iii) The integrand in the first integral on the right-hand side of (3.9) is a scalar multiple of $(\nabla u(x) \cdot (x - x^0) - 2u(x))^2$, and therefore vanishes if and only if u is a homogeneous function of degree 2 with respect to x^0 .
- (iv) The integrand in the first integral on the right-hand side of (3.9) is a scalar multiple of $(\nabla u(x) \cdot x - \frac{5}{2}u(x))^2$, and therefore vanishes if and only if u is a homogeneous function of degree $5/2$.

Proof. First, for each $u \in W_{w,loc}^{1,2}(\mathbf{R}^2)$, each $\alpha \in \mathbf{R}$ and a.e. $r \in (0, \delta)$ we obtain, setting $w_r(x) := u(x^0 + rx)$,

$$\begin{aligned}
\frac{d}{dr} \left(r^\alpha \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \right) &= \frac{d}{dr} \left(r^{\alpha+n-1} \int_{\partial B_1^+} \frac{1}{x_1^0 + rx_1} w_r^2 d\mathcal{H}^1 \right) \quad (3.12) \\
&= (\alpha + n - 1) r^{\alpha-1} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 - r^{\alpha+n-1} \int_{\partial B_1^+} \frac{x_1}{(x_1^0 + rx_1)^2} w_r^2 d\mathcal{H}^1 \\
&\quad + r^{\alpha+n-1} \int_{\partial B_1^+} \frac{2}{x_1^0 + rx_1} w_r \nabla u(x^0 + rx) \cdot x d\mathcal{H}^1 \\
&= (\alpha + n - 1) r^{\alpha-1} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 - r^{\alpha-1} \int_{\partial B_r^+(x^0)} \frac{x_1 - x_1^0}{(x_1)^2} u^2 d\mathcal{H}^1 \\
&\quad + r^\alpha \int_{\partial B_r^+(x^0)} \frac{2}{x_1} u \nabla u \cdot \nu d\mathcal{H}^1.
\end{aligned}$$

Suppose now that u is a variational solution of (3.1). For small positive κ and $\eta_\kappa(t) := \max(0, \min(1, \frac{r-t}{\kappa}))$, we take after approximation $\phi_\kappa(x) := \eta_\kappa(|x - x^0|)(x - x^0)$ as a test function in the definition of a variational solution, obtaining

$$\begin{aligned}
0 &= \int_{\Omega} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) (2\eta_\kappa(|x - x^0|) + \eta'_\kappa(|x - x^0|)|x - x^0|) dx \\
&\quad - \int_{\Omega} \frac{2}{x_1} \left[(\partial_1 u)^2 \left(\eta_\kappa(|x - x^0|) + \eta'_\kappa(|x - x^0|) \frac{(x_1 - x_1^0)^2}{|x - x^0|} \right) \right. \\
&\quad \left. + (\partial_2 u)^2 \left(\eta_\kappa(|x - x^0|) + \eta'_\kappa(|x - x^0|) \frac{(x_2 - x_2^0)^2}{|x - x^0|} \right) \right. \\
&\quad \left. + (\partial_1 u)(\partial_2 u) 2\eta'_\kappa(|x - x^0|) \frac{(x_1 - x_1^0)(x_2 - x_2^0)}{|x - x^0|} \right] dx \\
&\quad + \int_{\Omega} \left(- \frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 \eta_\kappa(|x - x^0|) + [(x_1 - x_1^0)x_2 \right. \\
&\quad \left. + (x_2 - x_2^0)x_1] \chi_{\{u>0\}} \eta_\kappa(|x - x^0|) \right) dx.
\end{aligned}$$

Passing to the limit as $\kappa \rightarrow 0$, we obtain for a.e. $r \in (0, \delta)$,

$$\begin{aligned} 0 = & 2 \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} dx - r \int_{\partial B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) d\mathcal{H}^1 \\ & + 2r \int_{\partial B_r^+(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1 \\ & + \int_{B_r^+(x^0)} \left(-\frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + [(x_1 - x_1^0)x_2 + (x_2 - x_2^0)x_1] \chi_{\{u>0\}} \right) dx. \end{aligned} \quad (3.13)$$

Observe that letting $\epsilon \rightarrow 0$ in

$$\int_{B_r^+(x^0)} \frac{1}{x_1} \nabla u \cdot \nabla \max(u - \epsilon, 0)^{1+\epsilon} dx = \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \max(u - \epsilon, 0)^{1+\epsilon} \nabla u \cdot \nu d\mathcal{H}^1$$

for a.e. $r \in (0, \delta)$, we obtain the integration by parts formula

$$\int_{B_r^+(x^0)} \frac{1}{x_1} |\nabla u|^2 dx = \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu d\mathcal{H}^1 \quad (3.14)$$

for a.e. $r \in (0, \delta)$.

Note that

$$\begin{aligned} (r^{-2} I(r))' = & -2r^{-3} \int_{B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) dx \\ & + r^{-2} \int_{\partial B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) d\mathcal{H}^1, \end{aligned}$$

so that by (3.13) and (3.14),

$$\begin{aligned} (r^{-2} I(r))' = & r^{-3} \left(2r \int_{\partial B_r^+(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1 - 2 \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu d\mathcal{H}^1 \right. \\ & \left. + \int_{B_r^+(x^0)} \left(-\frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + [(x_1 - x_1^0)x_2 + (x_2 - x_2^0)x_1] \chi_{\{u>0\}} \right) dx \right), \end{aligned} \quad (3.15)$$

Combining (3.15) and (3.12) with $\alpha = -3$ yields (3.8).

Moreover,

$$\begin{aligned} (r^{-3} I(r))' = & -3r^{-4} \int_{B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) dx \\ & + r^{-3} \int_{\partial B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) d\mathcal{H}^1. \end{aligned} \quad (3.16)$$

In the case $x_2^0 = 0$ we obtain from (3.16), using (3.13) and (3.14), that

$$\begin{aligned} (r^{-3} I(r))' = & r^{-4} \left(2r \int_{\partial B_r^+(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1 - 3 \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu d\mathcal{H}^1 \right. \\ & \left. + \int_{B_r^+(x^0)} \left(-\frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + (x_1 - x_1^0) x_2 \chi_{\{u>0\}} \right) dx \right), \end{aligned} \quad (3.17)$$

Combining (3.17) and (3.12) with $\alpha = -4$ yields (3.9). On the other hand, in the case $x_1^0 = 0$ we obtain from (3.16), using (3.13) and (3.14), that

$$(r^{-3}I(r))' = r^{-4} \left(2r \int_{\partial B_r^+(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1 - 4 \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu d\mathcal{H}^1 \right. \\ \left. + \int_{B_r^+(x^0)} (x_2 - x_2^0) x_1 \chi_{\{u>0\}} dx \right), \quad (3.18)$$

Combining (3.18) and (3.12) with $\alpha = -4$ yields (3.10).

Last, in the case $x_1^0 = x_2^0 = 0$, since

$$(r^{-4}I(r))' = -4r^{-5} \int_{B_r^+(0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) dx \\ + r^{-4} \int_{\partial B_r^+(0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) d\mathcal{H}^1,$$

we obtain from (3.13) and (3.14) that

$$(r^{-4}I(r))' = r^{-5} \left(2r \int_{\partial B_r^+(0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1 - 5 \int_{\partial B_r^+(0)} \frac{1}{x_1} u (\nabla u \cdot \nu) d\mathcal{H}^1 \right). \quad (3.19)$$

Combining (3.19) and (3.12) with $\alpha = -5$ yields (3.11). \square

Lemma 3.4 (Bernstein estimate). *In $\{u > 0\}$, the solution satisfies*

$$\Delta \left(\frac{|\nabla u|^2}{x_1} - x_1 x_2 \right) = 2 \sum_{i,j=1}^2 \frac{(\partial_{ij} u)^2}{x_1}.$$

Proof. Direct calculation. \square

Remark 3.5. Constructing barrier solutions it is therefore possible to verify $\frac{|\nabla u|^2}{x_1} - x_1 x_2 \leq 0$ for certain domains $\subset \{x_2 > 0\}$, certain Dirichlet boundary data and the *minimal solution* u (cf. [22]).

Definition 3.6 (Weak Solution). We define $u \in W_{w,\text{loc}}^{1,2}(\Omega)$ to be a *weak solution* of (3.1) if the following are satisfied: u is a *variational solution* of (3.1) and the topological free boundary $\partial\{u > 0\} \cap \Omega^\circ \cap \{x_2 \neq 0\}$ is locally a $C^{2,\alpha}$ -surface.

Remark 3.7. (i) It follows that in $\Omega^\circ \cap \{x_2 \neq 0\}$ the solution is a classical solution of (3.1). It follows also that $\partial\{u > 0\} \subset \{x_2 \geq 0\}$.

(ii) For any weak solution u of (3.1) such that

$$\frac{|\nabla u|^2}{x_1} \leq C x_1 |x_2| \quad \text{locally in } \Omega,$$

u is a variational solution of (3.1), $\chi_{\{u>0\}}$ is locally in $\Omega^\circ \cap \{x_2 > 0\}$ a function of bounded variation, and the total variation measure $|\nabla \chi_{\{u>0\}}|$ satisfies

$$\int_{B_r^+(x^0)} \sqrt{x_2^+} d|\nabla \chi_{\{u>0\}}| \leq C_1 \begin{cases} r^{\frac{3}{2}}, x_2^0 = 0 \\ r\sqrt{x_2^0}, x_2^0 > 0 \end{cases}$$

for all $B_r^+(x^0) \subset \subset \Omega$. The reason is that, integrating by parts,

$$\begin{aligned} 0 &= \int_{B_r^+(x^0) \cap \{u>0\}} \operatorname{div} \left(\frac{1}{x_1} \nabla u \right) \leq \int_{\partial B_r^+(x^0) \cap \{u>0\}} \frac{|\nabla u|}{x_1} d\mathcal{H}^1 \\ &\quad + \int_{B_r(x^0) \cap \{x_1=0\}} \frac{|\nabla u|}{x_1} d\mathcal{H}^1 - \int_{B_r^+(x^0) \cap \partial_{\text{red}}\{u>0\}} \frac{|\nabla u|}{x_1} d\mathcal{H}^1 \\ &\leq C_1 \left(\int_{\partial B_r^+(x^0)} \sqrt{x_2^+} d\mathcal{H}^1 + \int_{B_r(x^0) \cap \{x_1=0\}} \sqrt{x_2^+} d\mathcal{H}^1 \right) \\ &\quad - \int_{B_r^+(x^0) \cap \partial_{\text{red}}\{u>0\}} \sqrt{x_2^+} d\mathcal{H}^1. \end{aligned}$$

Lemma 3.8. *Let u be a variational solution of (3.1) and suppose that*

$$\frac{|\nabla u|^2}{x_1} \leq C x_1 |x_2| \quad \text{locally in } \Omega.$$

Then:

(i) *The limit $M^{\text{int}}(0+) = \lim_{r \rightarrow 0+} M^{\text{int}}(r)$ exists and is finite. If $x_2^0 = 0$, then the limit $M^{x_2}(0+) = \lim_{r \rightarrow 0+} M^{x_2}(r)$ exists and is finite. If $x_1^0 = 0$, then the limit $M^{x_1}(0+) = \lim_{r \rightarrow 0+} M^{x_1}(r)$ exists and is finite. If $x_1^0 = x_2^0 = 0$, then the limit $M^{x_1 x_2}(0+) = \lim_{r \rightarrow 0+} M^{x_1 x_2}(r)$ exists and is finite.*

(iii) *Let $x_1^0 > 0$, $x_2^0 > 0$ and $0 < r_m \rightarrow 0+$ as $m \rightarrow \infty$ be a sequence such that the blow-up sequence*

$$u_m(x) := u(x^0 + r_m x) / r_m \quad (3.20)$$

converges weakly in $W_{\text{loc}}^{1,2}(\mathbf{R}^2)$ to a blow-up limit u_0 . Then u_0 is a homogeneous function of degree 1, i.e. $u_0(\lambda x) = \lambda u_0(x)$.

Let $x_2^0 = 0$ and let $0 < r_m \rightarrow 0+$ as $m \rightarrow \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := u(x^0 + r_m x) / r_m^{\frac{3}{2}} \quad (3.21)$$

converges weakly in $W_{\text{loc}}^{1,2}(\mathbf{R}^2)$ to a blow-up limit u_0 . Then u_0 is a homogeneous function of degree $3/2$.

Let $x_1^0 = 0$ and let $0 < r_m \rightarrow 0+$ as $m \rightarrow \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := u(x^0 + r_m x) / r_m^2 \quad (3.22)$$

converges weakly in $W_{w,\text{loc}}^{1,2}(\mathbf{R}_+^2)$ to a blow-up limit u_0 . Then u_0 is a homogeneous function of degree 2.

Let $x_1^0 = x_2^0 = 0$ and let $0 < r_m \rightarrow 0+$ as $m \rightarrow \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := u(x^0 + r_m x)/r_m^{\frac{5}{2}} \quad (3.23)$$

converges weakly in $W_{w,\text{loc}}^{1,2}(\mathbf{R}_+^2)$ to a blow-up limit u_0 . Then u_0 is a homogeneous function of degree $5/2$.

(iii) Let u_m be one of the converging sequences in (ii). Then u_m converges strongly in $W_{w,\text{loc}}^{1,2}(\mathbf{R}_+^2)$ (strongly in $W_{\text{loc}}^{1,2}(\mathbf{R}^2)$ in the cases where $x_1^0 > 0$).

(iv) If $x_1^0 > 0$ and $x_2^0 \neq 0$, then

$$M^{\text{int}}(0+) = x_1^0 x_2^0 \lim_{r \rightarrow 0+} r^{-2} \int_{B_r^+(x^0)} \chi_{\{u>0\}} dx.$$

Moreover, $M^{\text{int}}(0+) = 0$ implies that $u_0 = 0$ in \mathbf{R}^2 for each blow-up limit u_0 of $u_m(x) = u(x^0 + r_m x)/r_m$.

If $x_1^0 > 0$ and $x_2^0 = 0$, then

$$M^{x_2}(0+) = x_1^0 \lim_{r \rightarrow 0+} r^{-3} \int_{B_r^+(x^0)} x_2 \chi_{\{u>0\}} dx.$$

If $x_1^0 = 0$ and $x_2^0 \neq 0$, then

$$M^{x_1}(0+) = x_2^0 \lim_{r \rightarrow 0+} r^{-3} \int_{B_r^+(x^0)} x_1 \chi_{\{u>0\}} dx.$$

Moreover, $M^{x_1}(0+) = 0$ implies that $u_0 = 0$ in \mathbf{R}_+^2 for each blow-up limit u_0 of $u_m(x) = u(x^0 + r_m x)/r_m^2$.

If $x_1^0 = x_2^0 = 0$, then

$$M^{x_1 x_2}(0+) = \lim_{r \rightarrow 0+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} dx.$$

Proof. (i) follows from the assumption

$$\frac{|\nabla u|^2}{x_1} \leq C x_1 |x_2| \quad \text{locally in } \Omega$$

together with Theorem 3.2.

(ii): For each $0 < \sigma < \infty$ the sequence u_m is in each case by assumption bounded in $C^{0,1}(B_\sigma^+)$ (bounded in $C^{0,1}(B_\sigma)$ in the case that $x_1^0 > 0$). For any $0 < \tau < \sigma < \infty$, we write the identities (3.8), (3.9), (3.8), (3.11) in integral form as

$$\begin{aligned} & 2 \int_\tau^\sigma r^{-2} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{u}{r} \right)^2 d\mathcal{H}^1 dr \\ &= M^{\text{int}}(\sigma) - M^{\text{int}}(\tau) - \int_\tau^\sigma K^{\text{int}}(r) dr \text{ in the case } x_1^0 > 0 \text{ and } x_2^0 > 0, \end{aligned} \quad (3.24)$$

$$\begin{aligned} & 2 \int_\tau^\sigma r^{-3} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{3}{2} \frac{u}{r} \right)^2 d\mathcal{H}^1 dr \\ &= M^{x_2}(\sigma) - M^{x_2}(\tau) - \int_\tau^\sigma K^{x_2}(r) dr \text{ in the case } x_1^0 > 0 \text{ and } x_2^0 = 0, \end{aligned} \quad (3.25)$$

$$\begin{aligned}
& 2 \int_{\tau}^{\sigma} r^{-3} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - 2 \frac{u}{r} \right)^2 d\mathcal{H}^1 dr \\
& = M^{x_1}(\sigma) - M^{x_1}(\tau) - \int_{\tau}^{\sigma} K^{x_1}(r) dr \text{ in the case } x_1^0 = 0 \text{ and } x_2^0 > 0, \quad (3.26)
\end{aligned}$$

$$\begin{aligned}
& 2 \int_{\tau}^{\sigma} r^{-4} \int_{\partial B_r^+(x^0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{5}{2} \frac{u}{r} \right)^2 d\mathcal{H}^1 dr \\
& = M^{x_1 x_2}(\sigma) - M^{x_1 x_2}(\tau) dr \text{ in the case } x_1^0 = x_2^0 = 0; \quad (3.27)
\end{aligned}$$

here K^{int} , K^{x_2} and K^{x_1} are defined by (3.24), (3.25) and (3.26), and they are all integrable.

It follows by rescaling in (3.24)-(3.27) that

$$\begin{aligned}
& 2 \int_{B_{\sigma}(0) \setminus B_{\tau}(0)} |x|^{-3} \frac{1}{x_1} (\nabla u_m(x) \cdot x - u_m(x))^2 dx \\
& \leq M^{int}(r_m \sigma) - M^{int}(r_m \tau) + \int_{r_m \tau}^{r_m \sigma} |K^{int}(r)| dr \rightarrow 0 \quad \text{as } m \rightarrow \infty, \\
& \text{in the case } x_1^0 > 0 \text{ and } x_2^0 > 0, \\
& 2 \int_{B_{\sigma}(0) \setminus B_{\tau}(0)} |x|^{-5} \frac{1}{x_1} \left(\nabla u_m(x) \cdot x - \frac{3}{2} u_m(x) \right)^2 dx \\
& \leq M^{x_2}(r_m \sigma) - M^{x_2}(r_m \tau) + \int_{r_m \tau}^{r_m \sigma} |K^{x_2}(r)| dr \rightarrow 0 \quad \text{as } m \rightarrow \infty, \\
& \text{in the case } x_1^0 > 0 \text{ and } x_2^0 = 0, \\
& 2 \int_{B_{\sigma}^+(0) \setminus B_{\tau}^+(0)} |x|^{-5} \frac{1}{x_1} (\nabla u_m(x) \cdot x - 2u_m(x))^2 dx \\
& \leq M^{x_1}(r_m \sigma) - M^{x_1}(r_m \tau) + \int_{r_m \tau}^{r_m \sigma} |K^{x_1}(r)| dr \rightarrow 0 \quad \text{as } m \rightarrow \infty, \\
& \text{in the case } x_1^0 = 0 \text{ and } x_2^0 > 0, \\
& 2 \int_{B_{\sigma}^+(0) \setminus B_{\tau}^+(0)} |x|^{-6} \frac{1}{x_1} \left(\nabla u_m(x) \cdot x - \frac{5}{2} u_m(x) \right)^2 dx \\
& \leq M^{x_1 x_2}(r_m \sigma) - M^{x_1 x_2}(r_m \tau) \rightarrow 0 \quad \text{as } m \rightarrow \infty \\
& \text{in the case } x_1^0 = x_2^0 = 0,
\end{aligned}$$

which yields the desired homogeneity of u_0 .

(iii): In order to show strong convergence of ∇u_m , it is in view of the weak L_w^2 -convergence of ∇u_m sufficient to prove convergence of the L_w^2 -norm.

Let $\delta := \text{dist}(x^0, \partial\Omega)/2$. Then, for each m , u_m is a variational solution of

$$\text{div} \left(\frac{\nabla u_m(x)}{(x^0 + r_m x)_1} \right) = 0 \quad \begin{cases} \text{in } B_{\delta/r_m} \cap \{u_m > 0\} \text{ in the case } x_1^0 > 0, \\ \text{in } B_{\delta/r_m}^+ \cap \{u_m > 0\} \text{ in the case } x_1^0 = 0. \end{cases} \quad (3.28)$$

Since u_m converges to u_0 locally uniformly, it follows from (3.28) that u_0 is harmonic in $\{u_0 > 0\}$ in the case $x_1^0 > 0$ and a solution of the equation

$$\operatorname{div} \left(\frac{1}{x_1} \nabla u_0 \right) = 0$$

in the case $x_1^0 = 0$. Also, using the uniform convergence, the continuity of u_0 and its solution property in $\{u_0 > 0\}$ we obtain as in the proof of (3.14) that

$$\begin{aligned} & o(1) + \int_{\mathbf{R}^2} \frac{1}{x_1^0} |\nabla u_m|^2 \eta \, dx \\ &= \int_{\mathbf{R}^2} \frac{1}{(x^0 + r_m x)_1} |\nabla u_m|^2 \eta \, dx = - \int_{\mathbf{R}^2} u_m \frac{1}{(x^0 + r_m x)_1} \nabla u_m \cdot \nabla \eta \, dx \\ &\rightarrow - \int_{\mathbf{R}^2} u_0 \frac{1}{x_1^0} \nabla u_0 \cdot \nabla \eta \, dx = \frac{1}{x_1^0} \int_{\mathbf{R}^2} |\nabla u_0|^2 \eta \, dx \text{ in the case } x_1^0 > 0 \text{ and that} \\ &\int_{\mathbf{R}_+^2} \frac{1}{x_1} |\nabla u_m|^2 \eta \, dx = - \int_{\mathbf{R}_+^2} u_m \frac{1}{x_1} \nabla u_m \cdot \nabla \eta \, dx \\ &\rightarrow - \int_{\mathbf{R}_+^2} u_0 \frac{1}{x_1} \nabla u_0 \cdot \nabla \eta \, dx = \int_{\mathbf{R}_+^2} \frac{1}{x_1} |\nabla u_0|^2 \eta \, dx \text{ in the case } x_1^0 = 0 \end{aligned}$$

as $m \rightarrow \infty$. It therefore follows that ∇u_m converges strongly in L_w^2 (and in L^2 if $x_1^0 > 0$) to ∇u_0 as $m \rightarrow \infty$.

(iv): Let us take a sequence $r_m \rightarrow 0+$ such that u_m defined in (3.20)-(3.23) converges weakly in $W_{w,\text{loc}}^{1,2}(\mathbf{R}_+^2)$ (weakly in $W_{\text{loc}}^{1,2}(\mathbf{R}^2)$ in the case $x_1^0 > 0$) to a function u_0 . Using (iii) and the homogeneity of u_0 , we obtain that

$$\begin{aligned} \lim_{m \rightarrow \infty} M^{\text{int}}(r_m) &= \frac{1}{x_1^0} \left(\int_{B_1} |\nabla u_0|^2 \, dx - \int_{\partial B_1} u_0^2 \, d\mathcal{H}^1 \right) \\ &\quad + \lim_{r \rightarrow 0+} r^{-2} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} \, dx \\ &= x_1^0 x_2^0 \lim_{r \rightarrow 0+} r^{-2} \int_{B_r^+(x^0)} \chi_{\{u>0\}} \, dx, \\ \lim_{m \rightarrow \infty} M^{x_2}(r_m) &= \frac{1}{x_1^0} \left(\int_{B_1} |\nabla u_0|^2 \, dx - \frac{3}{2} \int_{\partial B_1} u_0^2 \, d\mathcal{H}^1 \right) \\ &\quad + \lim_{r \rightarrow 0+} r^{-3} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} \, dx \\ &= x_1^0 \lim_{r \rightarrow 0+} r^{-3} \int_{B_r^+(x^0)} x_2 \chi_{\{u>0\}} \, dx, \\ \lim_{m \rightarrow \infty} M^{x_1}(r_m) &= \int_{B_1^+} \frac{1}{x_1} |\nabla u_0|^2 \, dx - 2 \int_{\partial B_1^+} \frac{1}{x_1} u_0^2 \, d\mathcal{H}^1 \\ &\quad + \lim_{r \rightarrow 0+} r^{-3} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} \, dx \end{aligned}$$

$$\begin{aligned}
&= x_2^0 \lim_{r \rightarrow 0+} r^{-3} \int_{B_r^+(x^0)} x_1 \chi_{\{u>0\}} dx, \\
\lim_{m \rightarrow \infty} M^{x_1 x_2}(r_m) &= \int_{B_1^+} \frac{1}{x_1} |\nabla u_0|^2 dx - \frac{5}{2} \int_{\partial B_1^+} \frac{1}{x_1} u_0^2 d\mathcal{H}^1 \\
&\quad + \lim_{r \rightarrow 0+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} dx \\
&= \lim_{r \rightarrow 0+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}} dx.
\end{aligned}$$

In the case $x_2^0 > 0$, $M^{int}(0+) \geq 0$ and $M^{x_1}(0+) \geq 0$, and equality implies that u_m converges to 0 in measure in \mathbf{R}_+^2 . \square

The next lemma will be useful in the characterization of blow-up limits in Proposition 3.10.

Lemma 3.9. *The Legendre function $y = P_{3/2}$ satisfies*

$$x \mapsto \frac{y'(x)}{y'(-x)} \text{ is strictly increasing on } (-1, 1).$$

Proof. It suffices to prove that

$$y''(x)y'(-x) + y''(-x)y'(x) > 0 \text{ in } (-1, 1).$$

Using the differential equation

$$(1 - x^2)y''(x) - 2xy'(x) + \frac{3}{2}y(x) = 0,$$

we obtain

$$y''(x)y'(-x) + y''(-x)y'(x) = -\frac{15}{4} \frac{1}{1-x^2} (y(x)y'(-x) + y(-x)y'(x)).$$

Therefore it is sufficient to prove that $f(x) = y(x)y'(-x) + y(-x)y'(x) < 0$ in $(-1, 1)$. As $f(x) \rightarrow -\infty$ for $|x| \rightarrow 1$, must have a maximum point in $(-1, 1)$. At the maximum point,

$$0 = f'(x) = \frac{2x}{1-x^2} (y(x)y'(-x) + y(-x)y'(x)),$$

implying that $x = 0$ and that

$$\begin{aligned}
\max f &= 2y(0)y'(0) = 3 \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{2})\Gamma(\frac{7}{4})} P_{\frac{1}{2}}(0) \\
&= 3 \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{4})\Gamma(\frac{7}{4})} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})} < 0
\end{aligned}$$

(see <http://functions.wolfram.com/07.07.20.0006.01>,
<http://functions.wolfram.com/07.07.03.0001.01>). \square

Proposition 3.10 (Characterization of blow-up Limits). *Let u be a variational solution of (3.1), and suppose that*

$$\frac{|\nabla u|^2}{x_1} \leq C x_1 |x_2| \quad \text{locally in } \Omega,$$

and that

$$\int_{B_r^+(x^0)} \sqrt{x_2^+} d|\nabla \chi_{\{u>0\}}| \leq C_1 \begin{cases} r^{\frac{3}{2}}, x_2^0 = 0 \\ r \sqrt{x_2^0}, x_2^0 > 0 \end{cases}$$

for all sufficiently small $r > 0$.

Then the following hold:

(i) In the case $x_1^0 > 0$ and $x_2^0 > 0$, the only possible blow-up limits of $u_m(x) = u(x^0 + r_m x)/r_m$ are

$$u_0(x) = x_1^0 \sqrt{x_2^0} \max(x \cdot e, 0) \quad \text{and} \quad u_0(x) = \gamma |x \cdot e|,$$

where e is a unit vector and γ is a nonnegative constant. In the case $u_0(x) = x_1^0 \sqrt{x_2^0} \max(x \cdot e, 0)$, the corresponding density is $M^{int}(0+) = x_1^0 x_2^0 \omega_2 / 2$, in the case $u_0(x) = \gamma |x \cdot e|$ with $\gamma > 0$ the density is $M^{int}(0+) = x_1^0 x_2^0 \omega_2$, while in the case $u_0 = 0$ the density has possible values $M^{int}(0+) \in \{0, x_1^0 x_2^0 \omega_2\}$.

(ii) In the case $x_1^0 > 0$ and $x_2^0 = 0$, the only possible blow-up limits are

$$u_0(\rho \sin \theta, \rho \cos \theta) = \frac{\sqrt{2} x_1^0}{3} \rho^{3/2} \cos\left(\frac{3}{2}\theta\right) \chi_{\{(\rho \sin \theta, \rho \cos \theta) : -\pi/3 < \theta < \pi/3\}},$$

with corresponding density

$$M^{x_2}(0+) = x_1^0 \int_{B_1} x_2 \chi_{\{(\rho \sin \theta, \rho \cos \theta) : -\pi/3 < \theta < \pi/3\}} dx,$$

and $u_0(x) = 0$, with possible values of the density

$$M^{x_2}(0+) \in \left\{ x_1^0 \int_{B_1} x_2^+ dx, x_1^0 \int_{B_1} x_2^- dx, 0 \right\}.$$

(iii) In the case $x_1^0 = 0$ and $x_2^0 > 0$, the only possible blow-up limits are

$$u_0(x) = \gamma x_1^2$$

with γ a nonnegative constant and corresponding density

$$M^{x_1}(0+) = x_2^0 \int_{B_1^+} x_1 dx,$$

and $u_0(x) = 0$, with possible values of the density

$$M^{x_1}(0+) \in \left\{ x_2^0 \int_{B_1^+} x_1 dx, 0 \right\}.$$

(iv) In the case $x_1^0 = x_2^0 = 0$, the only possible blow-up limits are

$$u_0(\rho \sin \theta, \rho \cos \theta) = r^{\frac{5}{2}} U_\ell(\theta)$$

with corresponding density

$$M^{x_1 x_2}(0+) = \int_{B_1^+ \cap \{(\rho \sin \theta, \rho \cos \theta) : P'_{3/2}(-\cos \theta) < 0\}} x_1 x_2 \, dx,$$

where $P_{3/2}$ is the Legendre function and U_ℓ is a unique function which is positive in $B_1^+ \cap \{P'_{3/2}(-\cos \theta) < 0\}$ (an angle of $\approx 114.799^\circ$ in the positive x_2 -direction) and zero else, and $u_0(x) = 0$, with possible values of the density

$$M^{x_1 x_2}(0+) \in \left\{ \int_{B_1^+} x_1 x_2^+ \, dx, \int_{B_1^+} x_1 x_2^- \, dx, 0 \right\}.$$

For U_ℓ we have the relations

$$\frac{5}{2}U_\ell(\theta) = c_0 \sin^2 \theta P'_{3/2}(\cos \theta), U'_\ell(\theta) = c_0 \frac{3}{2} \sin \theta P_{3/2}(\cos \theta)$$

with a unique positive constant c_0 .

Proof. Consider a blow-up sequence u_m as in Lemma 3.8, where $r_m \rightarrow 0+$, with blow-up limit u_0 . Because of the strong convergence of ∇u_m to ∇u_0 in L^2 and the compact embedding from BV into L^1 , u_0 is a homogeneous solution of

$$0 = \int_{\mathbf{R}^2} \frac{1}{x_1^0} \left(|\nabla u_0|^2 \operatorname{div} \phi - 2 \nabla u_0 D\phi \nabla u_0 \right) dx + x_1^0 x_2^0 \int_{\mathbf{R}^2} \chi_0 \operatorname{div} \phi \, dx \quad (3.29)$$

in the case $x_1^0 > 0$ and $x_2^0 > 0$,

$$0 = \int_{\mathbf{R}^2} \frac{1}{x_1^0} \left(|\nabla u_0|^2 \operatorname{div} \phi - 2 \nabla u_0 D\phi \nabla u_0 \right) dx + \int_{\mathbf{R}^2} \left(x_1^0 x_2 \chi_0 \operatorname{div} \phi + x_1^0 \chi_0 \phi_2 \right) dx \quad (3.30)$$

in the case $x_1^0 > 0$ and $x_2^0 = 0$,

$$0 = \int_{\mathbf{R}_+^2} \frac{1}{x_1} \left(|\nabla u_0|^2 \operatorname{div} \phi - \frac{1}{x_1} |\nabla u_0|^2 \phi_1 - 2 \nabla u_0 D\phi \nabla u_0 \right) dx + \int_{\mathbf{R}_+^2} \left(x_1 x_2^0 \chi_0 \operatorname{div} \phi + x_2^0 \chi_0 \phi_1 \right) dx \quad (3.31)$$

in the case $x_1^0 = 0$ and $x_2^0 > 0$;

$$0 = \int_{\mathbf{R}_+^2} \frac{1}{x_1} \left(|\nabla u_0|^2 \operatorname{div} \phi - \frac{1}{x_1} |\nabla u_0|^2 \phi_1 - 2 \nabla u_0 D\phi \nabla u_0 \right) dx + \int_{\mathbf{R}_+^2} \left(x_1 x_2 \chi_0 \operatorname{div} \phi + x_2 \chi_0 \phi_1 + x_1 \chi_0 \phi_2 \right) dx \quad (3.32)$$

in the case $x_1^0 = x_2^0 = 0$;

the formulas are valid for every $\phi = (\phi_1, \phi_2) \in C_0^1(\mathbf{R}^2; \mathbf{R}^2)$ in the case $x_1^0 > 0$ and for every $\phi = (\phi_1, \phi_2) \in C_0^1(\mathbf{R}^2; \mathbf{R}^2)$ such that $\phi_1 = 0$ on $\{x_1 = 0\}$ in the case $x_1^0 = 0$. Moreover χ_0 is the strong L_{loc}^1 -limit of $\chi_{\{u_m > 0\}}$ along a subsequence. The values of the function χ_0 are almost everywhere in $\{0, 1\}$, and the locally uniform convergence of u_m to u_0 implies that $\chi_0 = 1$ in $\{u_0 > 0\}$. Moreover χ_0 is constant

in each connected component of $\{u_0 = 0\}^\circ \setminus \{x_2 = 0\}$. In the case $u_0 = 0$, (3.29)-(3.32) show that χ_0 is constant in $\{x_2 \neq 0\}$ in the cases (3.30) and (3.32) and that χ_0 is constant in the cases (3.29) and (3.31). Its value may be either 0 or 1.

Let z be an arbitrary point in $\partial\{u_0 = 0\} \setminus \{0\}$. Consider first the case when $B_\delta(z) \cap \{u_0 > 0\}$ has exactly one connected component. Note that the normal to $\partial\{u_0 = 0\}$ has the constant value $\nu(z)$ in $B_\delta(z)$ for some $\delta > 0$. Plugging in $\phi(x) := \eta(x)\nu(z)$ into (3.29)-(3.32), where $\eta \in C_0^1(B_\delta(z))$ is arbitrary, and integrating by parts, it follows that

$$0 = \int_{\partial\{u_0 > 0\}} \left(-\frac{1}{x_1^0} |\nabla u_0|^2 + x_1^0 x_2^0 (1 - \bar{\chi}_0) \right) \eta d\mathcal{H}^1 \quad (3.33)$$

in the case $x_1^0 > 0$ and $x_2^0 > 0$,

$$0 = \int_{\partial\{u_0 > 0\}} \left(-\frac{1}{x_1^0} |\nabla u_0|^2 + x_1^0 x_2 (1 - \bar{\chi}_0) \right) \eta d\mathcal{H}^1 \quad (3.34)$$

in the case $x_1^0 > 0$ and $x_2^0 = 0$,

$$0 = \int_{\partial\{u_0 > 0\}} \left(-\frac{1}{x_1} |\nabla u_0|^2 + x_1 x_2^0 (1 - \bar{\chi}_0) \right) \eta d\mathcal{H}^1 \quad (3.35)$$

in the case $x_1^0 = 0$ and $x_2^0 > 0$,

$$0 = \int_{\partial\{u_0 > 0\}} \left(-\frac{1}{x_1} |\nabla u_0|^2 + x_1 x_2 (1 - \bar{\chi}_0) \right) \eta d\mathcal{H}^1 \quad (3.36)$$

in the case $x_1^0 = x_2^0 = 0$.

Here $\bar{\chi}_0$ denotes the constant value of χ_0 in $\{u_0 = 0\}^\circ$. Note that by Hopf's principle, $\nabla u_0 \cdot \nu \neq 0$ on $B_\delta(z) \cap \partial\{u_0 > 0\}$. In all cases it follows therefore that $\bar{\chi}_0 \neq 1$, and hence necessarily $\bar{\chi}_0 = 0$. We deduce from (3.33)-(3.36) that

$$\begin{aligned} |\nabla u_0|^2 &= (x_1^0)^2 x_2^0 \text{ on } \partial\{u_0 > 0\} \\ &\quad \text{in the case } x_1^0 > 0 \text{ and } x_2^0 > 0, \\ |\nabla u_0|^2 &= (x_1^0)^2 x_2 \text{ on } \partial\{u_0 > 0\} \\ &\quad \text{in the case } x_1^0 > 0 \text{ and } x_2^0 = 0, \\ |\nabla u_0|^2 &= x_1^2 x_2^0 \text{ on } \partial\{u_0 > 0\} \\ &\quad \text{in the case } x_1^0 = 0 \text{ and } x_2^0 > 0, \\ |\nabla u_0|^2 &= x_1^2 x_2 \text{ on } \partial\{u_0 > 0\} \\ &\quad \text{in the case } x_1^0 = x_2^0 = 0. \end{aligned}$$

Next, let us try to compute u_0 : In the cases where $x_1^0 > 0$, the homogeneity of u_0 and its harmonicity in $\{u_0 > 0\}$ imply the following: if $x_2^0 > 0$, then each connected component of $\{u_0 > 0\}$ is a half-plane passing through the origin. If $x_2^0 = 0$, then the fact that u_0 must be harmonic in $\{x_2 < 0\}$, implies that $\{u_0 > 0\}$ is a cone

with vertex at the origin and of opening angle 120° symmetric with respect to and containing $\{(0, t) : t > 0\}$.

In the cases where $x_1^0 = 0$, solving the resulting ODE leads to hypergeometric functions and is slightly awkward, so we will instead use, in each section of the unit disk where $u_0 > 0$, the velocity potential ϕ defined by

$$\partial_1 \phi = \frac{1}{x_1} \partial_2 u, \partial_2 \phi = -\frac{1}{x_1} \partial_1 u.$$

In the case $x_2^0 > 0$ we obtain that $\phi(\rho \sin \theta, \rho \cos \theta)$ is homogeneous of degree 1 and is on the unit circle given by a linear combination of $P_1(\cos \theta)$ and $\Re(Q_1(\cos \theta))$, where P_1 and Q_1 are the Legendre functions. Now $P_1(x) = x$ and $\Re Q_1$ is a strictly convex function with singularities at -1 and 1 , so that it is not possible that

$$\alpha P_1'(x) + (\Re Q_1)'(x) = \alpha P_1'(y) + (\Re Q_1)'(y) \text{ for } x \neq y \in (-1, 1).$$

It follows that there can be at most one free surface point of the solution $\alpha P_1(\cos \theta) + \Re Q_1(\cos \theta)$ in $(0, \pi)$, but then the solution would have at least one singularity in the interval $[0, \pi]$. Thus the only solution possible is $\sigma P_1(\cos \theta) = \sigma \cos \theta$, so that $\phi(x) = \sigma x_2$ and $u_0(x) = c x_1^2$, where c and σ are non-negative constants. The statement about the density follows as $\chi_0 = 1$ in $\{u_0 > 0\}$.

In the case $x_2^0 = 0$ we obtain that $\phi(\rho \sin \theta, \rho \cos \theta)$ is homogeneous of degree $3/2$ and is on the unit circle given by a linear combination of $P_{3/2}(\cos \theta)$ and $P_{3/2}(-\cos \theta)$, where $P_{3/2}$ is the Legendre function. It is well known that $P_{3/2}$ has only one singularity at -1 and that $P_{3/2}'$ has in $(-1, 1)$ a unique zero $z_0 \in (-1, 0)$. By Lemma 3.9 we obtain as in the last case that $\alpha P_{3/2}(\cos \theta) + \beta P_{3/2}(-\cos \theta)$ can have at most one free surface point in $(0, \pi)$. but then the solution would have at least one singularity in the interval $[0, \pi]$ unless $\beta = 0$. The fact that the singularity and the unique zero are both contained in $[-1, 0)$ implies therefore that either

$$\phi(\rho \sin \theta, \rho \cos \theta) = \sigma \rho^{3/2} P_{3/2}(\cos \theta) \text{ in } \{0 < \theta < \arccos(z_0)\}$$

or

$$\phi(\rho \sin \theta, \rho \cos \theta) = \sigma \rho^{3/2} P_{3/2}(-\cos \theta) \text{ in } \{\arccos(-z_0) < \theta < \pi\}.$$

However the free surface must not intersect $\{x_2 < 0\}$, so that we obtain that the only admissible solution is

$$\phi(\rho \sin \theta, \rho \cos \theta) = \sigma \rho^{3/2} P_{3/2}(-\cos \theta) \text{ in } \{\arccos(-z_0) < \theta < \pi\}$$

for some nonzero constant σ . Switching from the velocity potential back to u_0 we obtain the statement about u_0 as well as the density.

Last, consider the situation when the set $B_\delta(z) \cap \{u_0 > 0\}$ has two connected components. The computations of u_0 in the respective cases show that this is only possible for $x_1^0 > 0$ and $x_2^0 > 0$. The argument for (3.33) yields in this case that

the constant values of $|\nabla u_0|^2$ on either side of $\partial\{u_0 > 0\}$ are equal. This concludes the proof. \square

Lemma 3.11. *Let u be a weak solution of (3.1) such that $u = 0$ in $\{x_2 \leq 0\}$ and suppose that*

$$\frac{|\nabla u|^2}{x_1} \leq x_1 x_2^+ \quad \text{in } \Omega.$$

Then $x_2^0 = 0$, $x_1^0 > 0$ and $M^{x_2}(0+) = 0$ imply that $u \equiv 0$ in some open 2-dimensional ball containing x^0 , while $x_1^0 = x_2^0 = M^{x_1 x_2}(0+) = 0$ implies that $u \equiv 0$ in B_δ^+ for some $\delta > 0$.

Proof. Suppose towards a contradiction that $x^0 \in \partial\{u > 0\}$, and let us take a blow-up sequence

$$u_m(x) := u(x^0 + r_m x) / r_m^{3/2}$$

converging weakly in $W_{\text{loc}}^{1,2}(\mathbf{R}^2)$ to a blow-up limit u_0 in the case that $x_1^0 > 0$, and a blow-up sequence

$$u_m(x) := u(x^0 + r_m x) / r_m^{5/2}$$

converging weakly in $W_{w,\text{loc}}^{1,2}(\mathbf{R}_+^2)$ to a blow-up limit u_0 in the case that $x_1^0 = 0$. Proposition 3.10 shows that $u_0 = 0$ in \mathbf{R}^2 . Consequently,

$$0 \leftarrow \text{div} \left(\frac{1}{x_1^0 + r_m x_1} \nabla u_m \right) (B_2) \geq \int_{B_2 \cap \partial_{\text{red}}\{u_m > 0\}} \sqrt{x_2} d\mathcal{H}^1 \quad \text{in the case } x_1^0 > 0, \quad (3.37)$$

$$0 \leftarrow \text{div} \left(\frac{1}{x_1} \nabla u_m \right) (B_2^+) \geq \int_{B_2^+ \cap \partial_{\text{red}}\{u_m > 0\}} \sqrt{x_2} d\mathcal{H}^1 \quad \text{in the case } x_1^0 = 0$$

as $m \rightarrow \infty$. (Recall that $\text{div}(\frac{1}{x_1} \nabla u)$ is a nonnegative Radon measure in Ω .) On the other hand, there is at least one connected component V_m of $\{u_m > 0\}$ touching the origin and containing by the maximum principle a point $x^m \in \partial A$, where $A = (-1, 1) \times (0, 1)$ in the case $x_1^0 > 0$ and $A = (0, 1) \times (0, 1)$ in the case $x_1^0 = 0$. If $\max\{x_2 : x \in V_m \cap \partial A\} \not\rightarrow 0$ as $m \rightarrow \infty$, we immediately obtain a contradiction to (3.37). If $\max\{x_2 : x \in V_m \cap \partial A\} \rightarrow 0$, we use the free-boundary condition as well as $|\nabla u|^2/x_1^2 \leq x_2^+$ to obtain

$$0 = \text{div} \left(\frac{1}{x_1^0 + r_m x_1} \nabla u_m \right) (V_m \cap A) \leq \int_{V_m \cap \partial A} \sqrt{x_2} d\mathcal{H}^1 - \int_{A \cap \partial_{\text{red}} V_m} \sqrt{x_2} d\mathcal{H}^1$$

in the case $x_1^0 > 0$,

$$0 = \text{div} \left(\frac{1}{x_1} \nabla u_m \right) (V_m \cap A) \leq \int_{V_m \cap \partial A} \sqrt{x_2} d\mathcal{H}^1 - \int_{A \cap \partial_{\text{red}} V_m} \sqrt{x_2} d\mathcal{H}^1$$

in the case $x_1^0 = 0$.

However $\int_{V_m \cap \partial A} \sqrt{x_2} d\mathcal{H}^1$ is the unique minimiser of $\int_{\partial D} \sqrt{x_2} d\mathcal{H}^1$ with respect to all open sets D with $D = V_m$ on ∂A . So V_m cannot touch the origin, a contradiction. \square

Theorem 3.12 (Curve Case). *Let u be a weak solution of (3.1) satisfying*

$$\frac{|\nabla u|^2}{x_1} \leq C x_1 |x_2| \quad \text{locally in } \Omega,$$

and let $x^0 \in \Omega$ be such that $x_1^0 x_2^0 = 0$. Suppose in addition that $\partial\{u > 0\} \cap B_1^+(x^0)$ is in a neighborhood of x^0 a continuous injective curve $\sigma : I \rightarrow \mathbf{R}^2$ such that $\sigma = (\sigma_1, \sigma_2)$ and $\sigma(0) = x^0$. Then the following hold:

(i_1) Stokes corner: *If $x_1^0 > 0$, $x_2^0 = 0$ and*

$$M^{x_2}(0+) = x_1^0 \int_{B_1^+} x_2 \chi_{\{(\rho \sin \theta, \rho \cos \theta) : -\pi/3 < \theta < \pi/3\}} dx,$$

then (cf. Figure 4) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$ and, depending on the parametrization, either

$$\lim_{t \rightarrow 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}} \quad \text{and} \quad \lim_{t \rightarrow 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}},$$

or

$$\lim_{t \rightarrow 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \lim_{t \rightarrow 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}}.$$

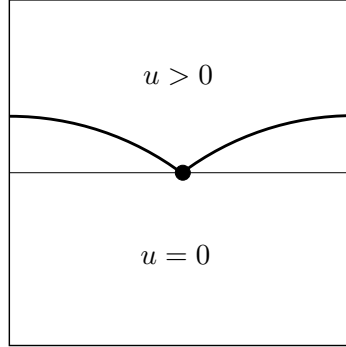
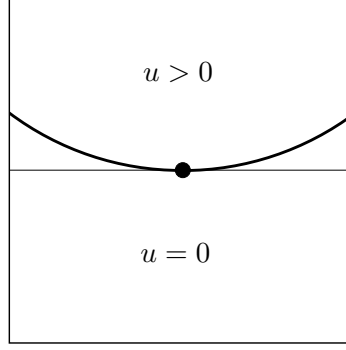


FIGURE 4. Stokes corner ($x_1^0 > 0, x_2^0 = 0$)

(i_2) *If $x_1^0 > 0$, $x_2^0 = 0$ and $M^{x_2}(0+) = x_1^0 \int_{B_1^+} x_2^+ dx$ or $M^{x_2}(0+) = x_1^0 \int_{B_1^+} x_2^- dx$, then (cf. Figure 5) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ changes sign at $t = 0$ and*

$$\lim_{t \rightarrow 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

FIGURE 5. Horizontal point ($x_1^0 > 0, x_2^0 = 0$)

(i₃) In the case $x_1^0 > 0, x_2^0 = 0$ and $M^{x_2}(0+) = 0$ —which is according to Lemma 3.11 not possible at all provided that $u = 0$ in $\{x_2 \leq 0\}$ and the sharp Bernstein inequality holds—, then $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ does not change its sign at $t = 0$, and

$$\lim_{t \rightarrow 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

(ii₁) If $x_2^0 > 0, x_1^0 = 0$ and $M^{x_1}(0+) = x_2^0 \int_{B_1^+} x_1 dx$, then (cf. Figures 6-8) $\sigma_2(t) \neq x_2^0$ in $(0, t_1)$ and

$$\lim_{t \rightarrow 0} \frac{\sigma_1(t)}{\sigma_2(t) - x_2^0} = 0,$$

or $\sigma_2(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_2 - x_2^0$ changes sign at $t = 0$ and

$$\lim_{t \rightarrow 0} \frac{\sigma_1(t)}{\sigma_2(t) - x_2^0} = 0.$$

(ii₂) The case $x_2^0 > 0, x_1^0 = 0$ and $M^{x_1}(0+) = 0$ is not possible.

(iii₁) Garabedian corner: If $x_1^0 = x_2^0 = 0$ and

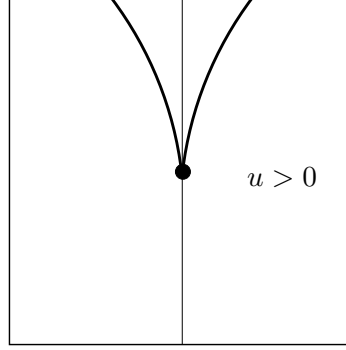
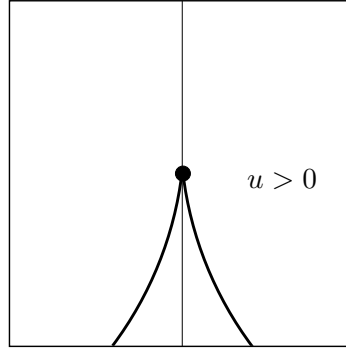
$$M^{x_1 x_2}(0+) = \int_{B_1^+ \cap \{P'_{3/2}(-\cos \theta) < 0\}} x_1 x_2 dx,$$

then (cf. Figure 9) $\sigma_1(t) \neq 0$ in $(0, t_1)$ and,

$$\lim_{t \rightarrow 0+} \frac{\sigma_2(t)}{\sigma_1(t)} = \tan(\pi/2 - \arccos(-z_0)).$$

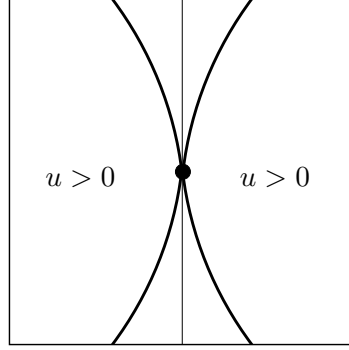
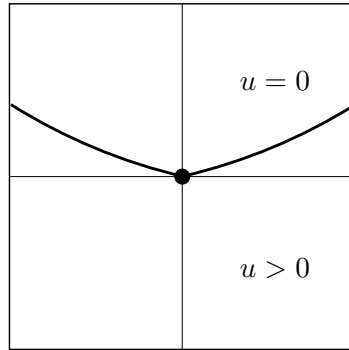
(iii₂) If $x_1^0 = x_2^0 = 0$ and

$$M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ dx \text{ or } M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^- dx,$$

FIGURE 6. Downward Vertical Cusp ($x_1^0 = 0, x_2^0 > 0$)FIGURE 7. Upward Vertical Cusp ($x_1^0 = 0, x_2^0 > 0$)

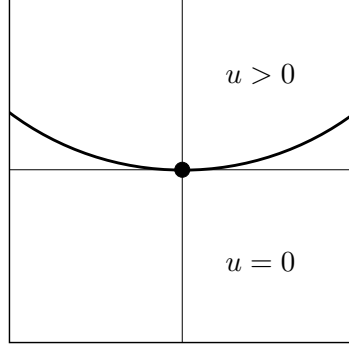
then (cf. Figure 10) $\sigma_1(t) \neq 0$ in $(0, t_1)$ and

$$\lim_{t \rightarrow 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0.$$

FIGURE 8. Double Vertical Cusp ($x_1^0 = 0, x_2^0 > 0$)FIGURE 9. Garabedian corner ($x_1^0 = x_2^0 = 0$)

(In the subsequent sections of the present paper we will analyze the precise asymptotics of the velocity field in the case $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ dx$.)

(iii₃) If $x_1^0 = x_2^0 = 0$ and $M^{x_1 x_2}(0+) = 0$ —which is according to Lemma 3.11 not possible at all provided that $u = 0$ in $\{x_2 \leq 0\}$ and the sharp Bernstein

FIGURE 10. Horizontal point ($x_1^0 = x_2^0 = 0$)

inequality holds—, then $\sigma_1(t) \neq 0$ in $(-t_1, t_1) \setminus \{0\}$, and

$$\lim_{t \rightarrow 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0.$$

Remark 3.13. Although we omit a proof in the present paper, a perturbation of the frequency formula in [21] (see [20]) can be used to prove that, if $x_1^0 > 0$ and $x_2^0 = 0$, then case $M^{x_2}(0+) = x_1^0 \int_{B_1^+} x_2^+ dx$ is not possible. Case (ii_1) seems possible as we have a nontrivial homogeneous solution. We do at present not have an existence proof for the cusps suggested here.

Proof. We prove the claimed results only case (iii), when $x_1^0 = x_2^0 = 0$, the analysis in the other cases being similar. For each $y = (y_1, y_2) \in \mathbf{R}^2$ with $y_1 \geq 0$ and $(y_1, y_2) \neq (0, 0)$, we define $\theta(y) \in [0, \pi]$ by the relation

$$(y_1, y_2) = (\rho(y) \sin \theta(y), \rho(y) \cos \theta(y)).$$

We now consider the set

$$\mathcal{L} = \{\theta_0 \in [0, \pi] : \text{there is } t_m \rightarrow 0 \text{ such that } \theta(\sigma(t_m)) \rightarrow \theta_0 \text{ as } m \rightarrow \infty\}.$$

Note that in fact $\mathcal{L} \subset [0, \pi/2]$, since the free boundary $\partial\{u > 0\}$ is contained in $\{x_2 \geq 0\}$.

We now claim that: *The set \mathcal{L} is a subset of $\{0, \theta^*, \pi/2\}$, where $\theta^* = \arccos(-z_0)$ is the angle corresponding to the Garabedian cone.*

Indeed, suppose towards a contradiction that a sequence $0 \neq t_m \rightarrow 0+, m \rightarrow \infty$

exists such that $\theta(\sigma(t_m)) \rightarrow \theta_0 \in \mathcal{L} \setminus \{0, \theta^*, \pi/2\}$, let $r_m := |\sigma(t_m)|$ and let

$$u_m(x) := \frac{u(r_m x)}{r_m^{5/2}}.$$

For each $\rho > 0$ such that $\tilde{B} := B_\rho(\sin \theta_0, \cos \theta_0)$ satisfies

$$\emptyset = \tilde{B} \cap (\{(\alpha, 0) : \alpha \in \mathbf{R}_+\} \cup \{(0, \alpha) : \alpha \in \mathbf{R}_+\} \cup \{(\alpha \sin \theta^*, \alpha \cos \theta^*) : \alpha \in \mathbf{R}_+\}),$$

we infer from the formula for the unique blow-up limit u_0 (see Theorem 3.10) that the convergence of measures

$$(\operatorname{div} (\frac{1}{x_1} \nabla u_m))(\tilde{B}) \rightarrow (\operatorname{div} (\frac{1}{x_1} \nabla u_0))(\tilde{B}) = 0 \text{ as } m \rightarrow \infty.$$

On the other hand,

$$\operatorname{div} (\frac{1}{x_1} \nabla u_m) = \sqrt{x_2} \mathcal{H}^1 \llcorner \partial\{u_m > 0\},$$

which implies, since $\tilde{B} \cap \partial\{u_m > 0\}$ contains a curve of length at least $2\rho - o(1)$, that

$$0 \leftarrow (\operatorname{div} (\frac{1}{x_1} \nabla u_m))(\tilde{B}) \geq c(\theta_0, \rho) \text{ as } m \rightarrow \infty,$$

where $c(\theta_0, \rho) > 0$, a contradiction. This proves the property claimed.

Now, a continuity argument yields that \mathcal{L} is a connected set. Consequently the limit

$$\ell = \lim_{t \rightarrow 0+} \theta(\sigma(t))$$

exists and is contained in the set $\{0, \theta^*, \pi/2\}$. In what follows, we identify the value of ℓ in terms of the value of $M^{x_1 x_2}(0+)$.

Suppose first that $M^{x_1 x_2}(0+) = \int_{B_1^+ \cap \{P_{3/2}(-\cos \theta) < 0\}} x_1 x_2 dx$. Then, by Proposition 3.10, the blow-up limit is

$$u_0(\rho \sin \theta, \rho \cos \theta) = r^{\frac{5}{2}} U_\ell(\theta).$$

Since $(\operatorname{div} (\frac{1}{x_1} \nabla u_0))(B_{1/100}(\sin \theta^*, \cos \theta^*)) > 0$, it follows that we cannot have $\ell \in \{0, \pi/2\}$, and therefore we must have $\ell = \theta^*$. This proves case (iii₁) of the Theorem.

Suppose now that $M^{x_1 x_2}(0+) \in \left\{ \int_{B_1^+} x_1 x_2^+ dx, \int_{B_1^+} x_1 x_2^- dx, 0 \right\}$. Then the blow-up limit is $u_0(x) = 0$. The same argument given earlier in the proof shows that $\ell \neq \theta^*$, so that necessarily $\ell \in \{0, \pi/2\}$. But then the formula in Lemma 3.8 that

$$M^{x_1 x_2}(0+) = \lim_{r \rightarrow 0+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u > 0\}} dx,$$

shows that $\ell = 0$ implies $M^{x_1 x_2}(0+) = 0$, while $\ell = \pi/2$ implies that $M^{x_1 x_2}(0+) \in \left\{ \int_{B_1^+} x_1 x_2^+ dx, \int_{B_1^+} x_1 x_2^- dx, 0 \right\}$. However, the possibility that $M^{x_1 x_2}(0+) = 0$ and $\ell = 0$ is ruled out by the argument in the proof of Lemma 3.11, even in the absence of the strict Bernstein condition. This proves the cases (iii₂) and (iii₃) of the Theorem. □

4. FREQUENCY FORMULA

From now on we will focus on the case $x_1^0 = x_2^0 = 0$, $u = 0$ in $\{x_2 \leq 0\}$ and $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ dx$, in which we will derive a precise asymptotic profile of the velocity.

Theorem 4.1 (Frequency Formula). *Let u be a variational solution of (3.1), and let $\delta := \text{dist}(0, \partial\Omega)/2$. Let, for any $r \in (0, \delta)$,*

$$D(r) = \frac{r \int_{B_r^+(0)} \frac{1}{x_1} |\nabla u|^2 dx}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1}$$

and

$$V(r) = \frac{r \int_{B_r^+(0)} x_1 x_2 (1 - \chi_{\{u>0\}}) dx}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1}.$$

Then the “frequency”

$$\begin{aligned} H(r) &= D(r) - V(r) \\ &= \frac{r \int_{B_r^+(0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 (\chi_{\{u>0\}} - 1) \right) dx}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1} \end{aligned}$$

satisfies for a.e. $r \in (0, \delta)$ the identities

$$\begin{aligned} &H'(r) \\ &= \frac{2}{r} \int_{\partial B_r^+(0)} \frac{1}{x_1} \left[\frac{r(\nabla u \cdot \nu)}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \right)^{1/2}} - D(r) \frac{u}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 \\ &\quad + \frac{2}{r} V^2(r) + \frac{2}{r} V(r) \left(H(r) - \frac{5}{2} \right) \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} &H'(r) \\ &= \frac{2}{r} \int_{\partial B_r^+(0)} \frac{1}{x_1} \left[\frac{r(\nabla u \cdot \nu)}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \right)^{1/2}} - H(r) \frac{u}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 \\ &\quad + \frac{2}{r} V(r) \left(H(r) - \frac{5}{2} \right). \end{aligned} \quad (4.2)$$

Proof. Note that, for all $r \in (0, \delta)$,

$$H(r) = \frac{r^{-4} I(r) - \int_{B_1^+} x_1 x_2 dx}{r^{-5} J(r)}. \quad (4.3)$$

Hence, for a.e. $r \in (0, \delta)$,

$$H'(r) = \frac{(r^{-4} I(r))'}{r^{-5} J(r)} - \frac{(r^{-4} I(r) - \int_{B_1^+} x_1 x_2 dx)}{r^{-5} J(r)} \frac{(r^{-5} J(r))'}{r^{-5} J(r)},$$

Using the identities (3.19) and (3.12) with $\alpha = -5$, we therefore obtain that, for a.e. $r \in (0, \delta)$,

$$\begin{aligned}
H'(r) &= \frac{\left(2r \int_{\partial B_r^+(0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1 - 5 \int_{\partial B_r^+(0)} \frac{1}{x_1} u (\nabla u \cdot \nu) d\mathcal{H}^1\right)}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1} \\
&\quad - (D(r) - V(r)) \frac{1}{r} \frac{\left(2r \int_{\partial B_r^+(0)} \frac{1}{x_1} u (\nabla u \cdot \nu) d\mathcal{H}^1 - 5 \int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1\right)}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1} \\
&= \frac{2}{r} \left(\frac{r^2 \int_{\partial B_r^+(0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 d\mathcal{H}^1}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1} - \frac{5}{2} D(r) \right) \\
&\quad - \frac{2}{r} (D(r) - V(r)) \left(D(r) - \frac{5}{2} \right), \tag{4.4}
\end{aligned}$$

where we have also used the fact, which follows from (3.14), that

$$D(r) = \frac{r \int_{\partial B_r^+(0)} \frac{1}{x_1} u (\nabla u \cdot \nu) d\mathcal{H}^1}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1}. \tag{4.5}$$

Identity (4.1) now follows by merely rearranging (4.4), making use again of (4.5) and the fact that $D(r) = V(r) + H(r)$.

Since (4.1) holds, it follows by inspection that (4.2) holds if and only if

$$\begin{aligned}
&\int_{\partial B_r^+(0)} \frac{1}{x_1} [r(\nabla u \cdot \nu) - D(r)u]^2 d\mathcal{H}^1 + V^2(r) \int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \\
&= \int_{\partial B_r^+(0)} \frac{1}{x_1} [r(\nabla u \cdot \nu) - H(r)u]^2 d\mathcal{H}^1. \tag{4.6}
\end{aligned}$$

However, (4.6) is easily verified as a consequence of (4.5) and the fact that $D(r) = H(r) + V(r)$. In conclusion, identity (4.2) also holds. \square

Theorem 4.2. *Let u be a variational solution of (3.1) such that $u = 0$ in $\{x_2 \leq 0\}$, let $x^0 = 0$, suppose that $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ dx$, and let $\delta := \text{dist}(0, \partial\Omega)/2$. Then the following hold:*

- (i) $H(r) \geq \frac{5}{2}$ for all $r \in (0, \delta)$.
- (ii) The function $r \mapsto r^{-5}J(r)$ is nondecreasing on $(0, \delta)$.
- (iii) The function H is nondecreasing on $(0, \delta)$, and has a right limit $H(0+)$, where $H(0+) \geq 5/2$.
- (iv) $r \mapsto \frac{1}{r}V^2(r) \in L^1(0, \delta)$.

Proof. (i) The monotonicity, which follows from Theorem 3.2, of the function $M^{x_1 x_2}$ ensures that, for all $r \in (0, \delta)$,

$$r^{-4}I(r) - \frac{5}{2}r^{-5}J(r) \geq \int_{B_1^+} x_1 x_2^+ dx. \tag{4.7}$$

Using (4.3), the above inequality may be rearranged in the form of the claimed result.

(ii) Plugging $\alpha = -5$ into (3.12), using also (3.14), and then (4.7), we obtain, for a.e. $r \in (0, \delta)$,

$$\begin{aligned} (r^{-5}J(r))' &= \frac{2}{r} \left(r^{-4} \int_{B_r^+(0)} \frac{1}{x_1} |\nabla u|^2 dx - \frac{5}{2} r^{-5} \int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 d\mathcal{H}^1 \right) \\ &\geq 2r^{-5} \int_{B_r^+(0)} x_1 x_2 (1 - \chi_{\{u>0\}}) dx \geq 0, \end{aligned}$$

which implies the claimed result.

(iii) The monotonicity of H on $(0, \delta)$ is a consequence of (4.1) and (i). The remaining part of the claim is immediate.

(iv) The claimed result follows from (4.1) and (iii). \square

5. BLOW-UP LIMITS

The Frequency Formula allows passing to blow-up limits.

Proposition 5.1. *Let u be a variational solution of (3.1) such that $u = 0$ in $\{x_2 \leq 0\}$, let $x^0 = 0$, and suppose that $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ dx$. Then:*

(i) *There exist $\lim_{r \rightarrow 0+} V(r) = 0$ and $\lim_{r \rightarrow 0+} D(r) = H(0+)$.*

(ii) *For any sequence $r_m \rightarrow 0+$ as $m \rightarrow \infty$, the sequence*

$$v_m(x) := \frac{u(r_m x)}{\sqrt{r_m^{-1} \int_{\partial B_{r_m}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1}} \quad (5.1)$$

is bounded in $W_w^{1,2}(B_1^+)$.

(iii) *For any sequence $r_m \rightarrow 0+$ as $m \rightarrow \infty$ such that the sequence v_m in (5.1) converges weakly in $W_w^{1,2}(B_1^+)$ to a blow-up limit v_0 , the function v_0 is homogeneous of degree $H(0+)$ in B_1^+ , and satisfies*

$$v_0 \geq 0 \text{ in } B_1, v_0 \equiv 0 \text{ in } B_1^+ \cap \{x_2 \leq 0\} \text{ and } \int_{\partial B_1^+} \frac{1}{x_1} v_0^2 d\mathcal{H}^1 = 1.$$

Proof. We first prove that, for any sequence $r_m \rightarrow 0+$, the sequence v_m defined in (5.1) satisfies, for every $0 < \tau < \sigma < 1$,

$$\int_{B_\sigma^+ \setminus B_\tau^+} \frac{1}{x_1} |x|^{-5} [\nabla v_m(x) \cdot x - H(0+) v_m(x)]^2 dx \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (5.2)$$

Indeed, for any such τ and σ , it follows by scaling from (4.2) that, for every m such that $r_m < \delta$,

$$\begin{aligned} &\int_\tau^\sigma \frac{2}{r} \int_{\partial B_r^+} \frac{1}{x_1} \left[\frac{r(\nabla v_m \cdot \nu)}{\left(\int_{\partial B_r^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \right)^{1/2}} - H(r_m r) \frac{v_m}{\left(\int_{\partial B_r^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 dr \\ &\leq H(r_m \sigma) - H(r_m \tau) \rightarrow 0 \quad \text{as } m \rightarrow \infty, \end{aligned}$$

as a consequence of Theorem 4.2 (iii). The above implies that

$$\begin{aligned} \int_{\tau}^{\sigma} \frac{2}{r} \int_{\partial B_r^+} \frac{1}{x_1} \left[\frac{r(\nabla v_m \cdot \nu)}{\left(\int_{\partial B_r^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \right)^{1/2}} - H(0+) \frac{v_m}{\left(\int_{\partial B_r^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 dr \\ \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned} \quad (5.3)$$

Now note that, for every $r \in (\tau, \sigma) \subset (0, 1)$ and all m as before, it follows by using Theorem 4.2 (ii), that

$$\int_{\partial B_r^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 = \frac{\int_{\partial B_{r_m}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1}{\int_{\partial B_{r_m}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1} \leq r^5 \leq 1.$$

Therefore (5.2) follows from (5.3), which proves our claim. Let us also recall (4.5).

We can now prove all parts of the Proposition.

(i) Suppose towards a contradiction that (i) is not true. Let $s_m \rightarrow 0$ be such that the sequence $V(s_m)$ is bounded away from 0. It is a consequence of Theorem 4.2(iv) that

$$\min_{r \in [s_m, 2s_m]} V(r) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Let $t_m \in [s_m, 2s_m]$ be such that $V(t_m) \rightarrow 0$ as $m \rightarrow \infty$. For the choice $r_m := t_m$ for every m , the sequence v_m given by (5.1) satisfies (5.2). The fact that $V(r_m) \rightarrow 0$ implies that $D(r_m)$ is bounded, and hence that v_m is bounded in $W_w^{1,2}(B_1^+)$. Let v_0 be any weak limit of v_m along a subsequence. Note that by the compact embedding $W_w^{1,2}(B_1^+) \hookrightarrow L^2(\partial B_1^+)$, v_0 has norm 1 on $L_w^2(\partial B_1^+)$, since this is true for v_m for all m . It follows from (5.2) that v_0 is homogeneous of degree $H(0+)$. Note that, by using Theorem 4.2 (ii),

$$\begin{aligned} V(s_m) &= \frac{s_m^{-4} \int_{B_{s_m}^+} x_1 x_2 (1 - \chi_{\{u>0\}}) dx}{s_m^{-5} \int_{\partial B_{s_m}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1} \\ &\leq \frac{s_m^{-4} \int_{B_{r_m}^+} x_1 x_2 (1 - \chi_{\{u>0\}}) dx}{(r_m/2)^{-5} \int_{\partial B_{r_m/2}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1} \\ &\leq \frac{1}{2} \frac{\int_{\partial B_{r_m}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1}{\int_{\partial B_{r_m/2}^+} \frac{1}{x_1} u^2 d\mathcal{H}^1} V(r_m) \\ &= \frac{1}{2 \int_{\partial B_{1/2}^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1} V(r_m). \end{aligned} \quad (5.4)$$

Since, at least along a subsequence,

$$\int_{\partial B_{1/2}^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \rightarrow \int_{\partial B_{1/2}^+} \frac{1}{x_1} v_0^2 d\mathcal{H}^1 > 0,$$

(5.4) leads to a contradiction. It follows that indeed $V(r) \rightarrow 0$ as $r \rightarrow 0+$. This implies that $D(r) \rightarrow H(0+)$.

(ii) Let r_m be an arbitrary sequence with $r_m \rightarrow 0+$. The boundedness of the sequence v_m in $W_w^{1,2}(B_1)$ is equivalent to the boundedness of $D(r_m)$, which is true by (i).

(iii) Let $r_m \rightarrow 0+$ be an arbitrary sequence such that v_m converges weakly to v_0 . The homogeneity degree $H(0+)$ of v_0 follows directly from (5.2). The fact that $\int_{\partial B_1^+} \frac{1}{x_1} v_0^2 d\mathcal{H}^1 = 1$ is a consequence of $\int_{\partial B_1^+} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 = 1$ for all m , and the remaining claims of the Proposition are obvious. The homogeneity of v_0 , together with the fact that v_0 belongs to $W_w^{1,2}(B_1^+)$, imply (in two dimensions) that v_0 is continuous. \square

6. CONCENTRATION COMPACTNESS

In the present section we will prove a concentration compactness result which allows us to preserve variational solutions in the blow-up limit at degenerate points and excludes concentration. In order to do so we combine the concentration compactness result of J.-M. Delort [8] with information gained by our Frequency Formula. In addition, we obtain strong convergence of our blow-up sequence which is necessary in order to prove our main theorems.

Theorem 6.1. *Let u be a variational solution of (3.1) such that $u = 0$ in $x_2 \leq 0$ and $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ dx$. Let $r_m \rightarrow 0+$ be such that the sequence v_m given by (5.1) converges weakly to v_0 in $W_w^{1,2}(B_1^+)$. Then v_m converges to v_0 strongly in $W_{w,\text{loc}}^{1,2}(B_1^+ \setminus \{0\})$, v_0 is continuous on B_1^+ and $\text{div}(\frac{1}{x_1} \nabla v_0)$ is a nonnegative Radon measure satisfying $v_0 \text{div}(\frac{1}{x_1} \nabla v_0) = 0$ in the sense of Radon measures in B_1^+ .*

Proof. Note first that the homogeneity of v_0 given by Proposition 5.1, together with the fact that v_0 belongs to $W_w^{1,2}(B_1^+)$, imply that v_0 is continuous.

Let σ and τ with $0 < \tau < \sigma < 1$ be arbitrary. We know that $\text{div}(\frac{1}{x_1} \nabla v_m) \geq 0$ and $\text{div}(\frac{1}{x_1} \nabla v_m)(B_{(\sigma+1)/2}^+) \leq C_1$ for all m . We regularize each v_m to

$$\tilde{v}_m := v_m * \phi_m \in C^\infty(B_1^+),$$

where ϕ_m is a standard mollifier such that

$$\text{div}(\frac{1}{x_1} \nabla \tilde{v}_m) \geq 0, \quad \int_{B_\sigma^+} \text{div}(\frac{1}{x_1} \nabla \tilde{v}_m) \leq C_2 < +\infty \quad \text{for all } m,$$

and

$$\left\| \frac{1}{x_1} (\nabla v_m - \nabla \tilde{v}_m) \right\|_{L^2(B_\sigma^+)} + \|v_m - \tilde{v}_m\|_{L^2(B_\sigma^+)} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Let us now consider the velocity field in three dimensions

$$V^m(X, Y, Z) := \left(-\frac{1}{x_1} \partial_2 \tilde{v}_m \cos \vartheta, -\frac{1}{x_1} \partial_2 \tilde{v}_m \sin \vartheta, \frac{1}{x_1} \partial_1 \tilde{v}_m \right),$$

where $(X, Y, Z) = (x_1 \cos \vartheta, x_1 \sin \vartheta, x_2)$, as well as their weak limit

$$V(X, Y, Z) := \left(-\frac{1}{x_1} \partial_2 \tilde{v} \cos \vartheta, -\frac{1}{x_1} \partial_2 \tilde{v} \sin \vartheta, \frac{1}{x_1} \partial_1 \tilde{v} \right).$$

We have that V^m is divergence free and satisfies

$$\operatorname{curl} V^m = \omega^m = (-\sin \vartheta, \cos \vartheta, 0) \alpha^m \text{ in } B_2(0)$$

with a non-negative function α^m that is bounded in $L^1(B_\sigma)$. It follows that

$$\begin{aligned} V_1^m &= \Delta_{m1}^{-1} \partial_Z \omega_2^m \\ V_2^m &= -\Delta_{m2}^{-1} \partial_Z \omega_1^m \\ V_3^m &= \Delta_{m3}^{-1} (\partial_Y \omega_1^m - \partial_X \omega_2^m), \end{aligned}$$

where Δ_{mi}^{-1} is the inverse of the three dimensional Laplace operator with averaged Dirichlet boundary data V_i^m , more precisely

$$\Delta_{mi}^{-1} f = \frac{2}{1-\sigma} \int_\sigma^{\frac{1+\sigma}{2}} \int_{B_R} G_R f \, dx \, dR + \frac{2}{1-\sigma} \int_\sigma^{\frac{1+\sigma}{2}} \int_{\partial B_R} V_i^m \nabla G_R \cdot \nu \, d\mathcal{H}^2,$$

where G_R is Green's function with respect to the Laplace operator in B_R . From the proof of [8, Proposition 3.2], where [8, (3.6)] holds with v_i^ε replaced by V_i^m and ω_i^ε replaced by ω_i^m but the remainder terms w_i^ε given by Greens formula in B_σ , we infer that

$$\begin{aligned} V_1^m V_3^m &\rightharpoonup V_1 V_3 \text{ weakly in } L_{\text{loc}}^2(B_\sigma), \\ V_2^m V_3^m &\rightharpoonup V_2 V_3 \text{ weakly in } L_{\text{loc}}^2(B_\sigma), \\ (V_1^m)^2 + (V_2^m)^2 - (V_3^m)^2 &\rightharpoonup (V_1)^2 + (V_2)^2 - (V_3)^2 \text{ weakly in } L_{\text{loc}}^2(B_\sigma); \end{aligned}$$

note that as in [8] the remainder terms converge strongly in $L_{\text{loc}}^2(B_\sigma)$.

It follows that

$$\frac{1}{x_1} \partial_1 v_m \partial_2 v_m \rightarrow \frac{1}{x_1} \partial_1 v_0 \partial_2 v_0 \quad (6.1)$$

and

$$\frac{1}{x_1} ((\partial_1 v_m)^2 - (\partial_2 v_m)^2) \rightarrow \frac{1}{x_1} ((\partial_1 v_0)^2 - (\partial_2 v_0)^2)$$

in the sense of distributions on B_σ^+ as $m \rightarrow \infty$. Let us remark that in contrast to the true two-dimensional problem, this alone would *not* allow us to pass to the limit in the domain variation formula for v_m !

Observe now that (5.2) shows that

$$\nabla v_m(x) \cdot x - H(0+) v_m(x) \rightarrow 0$$

strongly in $L_w^2(B_\sigma^+ \setminus B_\tau^+)$ as $m \rightarrow \infty$. It follows that

$$\partial_1 v_m x_1 + \partial_2 v_m x_2 \rightarrow \partial_1 v_0 x_1 + \partial_2 v_0 x_2$$

strongly in $L_w^2(B_\sigma^+ \setminus B_\tau^+)$ as $m \rightarrow \infty$. But then

$$\begin{aligned} & \int_{B_\sigma^+ \setminus B_\tau^+} \frac{1}{x_1} (\partial_1 v_m \partial_1 v_m x_1 + \partial_1 v_m \partial_2 v_m x_2) \eta \, dx \\ & \rightarrow \int_{B_\sigma^+ \setminus B_\tau^+} \frac{1}{x_1} (\partial_1 v_0 \partial_1 v_0 x_1 + \partial_1 v_0 \partial_2 v_0 x_2) \eta \, dx \end{aligned}$$

for each $\eta \in C_0^0(B_\sigma^+ \setminus \overline{B_\tau^+})$ as $m \rightarrow \infty$. Using (6.1), we obtain that

$$\int_{B_\sigma^+ \setminus B_\tau^+} (\partial_1 v_m)^2 \eta \, dx \rightarrow \int_{B_\sigma^+ \setminus B_\tau^+} (\partial_1 v_0)^2 \eta \, dx$$

for each $0 \leq \eta \in C_0^0(B_\sigma^+ \setminus \overline{B_\tau^+})$ as $m \rightarrow \infty$. Using once more (6.1) yields that ∇v_m converges strongly in $L_{w,\text{loc}}^2(B_\sigma^+ \setminus \overline{B_\tau^+})$. Since σ and τ with $0 < \tau < \sigma < 1$ were arbitrary, it follows that ∇v_m converges to ∇v_0 strongly in $L_{w,\text{loc}}^2(B_1^+ \setminus \{0\})$.

As a consequence of the strong convergence, we see that

$$\int_{B_1^+} \frac{1}{x_1} \nabla(\eta v_0) \cdot \nabla v_0 = 0 \quad \text{for all } \eta \in C_0^1(B_1^+ \setminus \{0\}).$$

Combined with the fact that $v_0 = 0$ in $B_1^+ \cap \{x_2 \leq 0\}$, this proves that $v_0 \Delta v_0 = 0$ in the sense of Radon measures on B_1^+ . \square

7. DEGENERATE POINTS

Theorem 7.1. *Let u be a weak solution of (3.1) such that $u = 0$ in $x_2 \leq 0$ and $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ \, dx$, let the free boundary $\partial\{u > 0\} \cap B_1^+$ be a continuous injective curve $\sigma = (\sigma_1, \sigma_2)$ such that $\sigma(0) = 0$. Then $\sigma_1(t) \neq 0$ in $[0, t_1) \setminus \{0\}$,*

$$\lim_{t \rightarrow 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0$$

and

$$\frac{u(rx)}{\sqrt{r^{-1} \int_{\partial B_r^+(0)} u^2 \, d\mathcal{H}^1}} \rightarrow \frac{x_1^2 x_2}{\sqrt{\int_{\partial B_1^+(0)} x_1^4 x_2^2 \, d\mathcal{H}^1}} \quad \text{as } r \rightarrow 0+,$$

strongly in $W_{w,\text{loc}}^{1,2}(B_1^+ \setminus \{0\})$ and weakly in $W^{1,2}(B_1^+)$. Moreover,

$$\begin{aligned} & \frac{u(rx)}{r^\alpha} \rightarrow 0 \text{ in } L_w^2(B_1^+) \text{ for } \alpha \in (0, 2) \text{ and} \\ & \frac{u(rx)}{r^\alpha} \text{ is unbounded in } L_w^2(B_1^+) \text{ for } \alpha > 2. \end{aligned}$$

Proof. Let $r_m \rightarrow 0+$ be an arbitrary sequence such that the sequence v_m given by (5.1) converges weakly in $W_w^{1,2}(B_1^+)$ to a limit v_0 . By Proposition 5.1 (iii) and Theorem 6.1, $v_0 \not\equiv 0$, v_0 is homogeneous of degree $H(0+) \geq 5/2$, v_0 is continuous, $v_0 \geq 0$ and $v_0 \equiv 0$ on $\{x_1 = 0\}$ and in $\{x_2 \leq 0\}$, $v_0 \operatorname{div}(\frac{1}{x_1} \nabla v_0) = 0$ in B_1^+ as a Radon measure, and the convergence of v_m to v_0 is strong in $W_{w,\text{loc}}^{1,2}(B_1^+ \setminus \{0\})$.

Moreover, the strong convergence of v_m and the fact proved in Proposition 5.1 (i) that $V(r_m) \rightarrow 0$ as $m \rightarrow \infty$ imply that

$$0 = \int_{\mathbf{R}^2} \left(\frac{1}{x_1} |\nabla v_0|^2 \operatorname{div} \phi - 2 \nabla u_0 D\phi \nabla v_0 \right)$$

for every $\phi \in C_0^1(\{x_1 > 0\} \cap \{x_2 > 0\}; \mathbf{R}^2)$, so that even an analysis in the case of $\{u = 0\}$ consisting of infinitely many disconnected components (similar to that in [21]) would be possible in principle. However the structure here is more complicated. For that reason we confine ourselves to the assumed injective curve case.

As in the proof of Proposition 3.10, we will use in each section of the unit disk where $v_0 > 0$ the velocity potential ϕ defined by

$$\partial_1 \phi = \frac{1}{x_1} \partial_2 v_0, \partial_2 \phi = -\frac{1}{x_1} \partial_1 v_0.$$

We obtain that $\phi(\rho \sin \theta, \rho \cos \theta)$ is homogeneous of degree $m = H(0+) \geq 5/2$ and is on the unit circle given by a linear combination $f(\cos \theta) = \alpha P_m(\cos \theta) + \beta P_m(-\cos \theta)$, in the case that the Legendre function P_m and the function $P_m(-x)$ are linearly independent, and $f(\cos \theta) = \alpha P_m(\cos \theta) + \beta \Re(Q_m(\cos \theta))$ in the case the Legendre function P_m and the function $P_m(-x)$ are linearly dependent. Moreover $(1, 0)$ is a free boundary point of v_0 so that $f'(0) = 0$, which implies $\alpha = \beta$ in the case of linear independence.

On the other hand, Theorem 3.12 (ii) implies that for any ball $\tilde{B} \subset\subset B_1^+ \cap \{x_2 > 0\}$, $v_r = \frac{u(rx)}{\sqrt{r^{-1} \int_{\partial B_r^+(0)} u^2 d\mathcal{H}^1}} > 0$ in \tilde{B} . Consequently $\operatorname{div}(\frac{1}{x_1} \nabla v_0) = 0$ in $\{x_1 > 0\} \cap \{x_2 > 0\}$. However, if there is a free boundary point x in $(0, 1) \times (0, 1)$ then by homogeneity the half line connecting that point to the origin consists of free boundary points, so that $(\operatorname{div}(\frac{1}{x_1} \nabla v_0))(B_\delta(x)) > 0$ for each $\delta > 0$, a contradiction. Thus $\alpha P'_m + \beta Q'_m$ must be either strictly positive or strictly negative in $(0, 1)$.

In the case $f(\cos \theta) = \alpha(P_m(\cos \theta) + P_m(-\cos \theta))$ we obtain now a contradiction to the fact that P_m is bounded at 1 and has a singularity at -1 .

In the case that P_m is an even function, we obtain from $P'_m(0) = mP_{m-1}(0) = \frac{m\sqrt{\pi}}{\Gamma(\frac{2-m}{2})\Gamma(\frac{m-1}{2}+1)}$ and $Q'_m(0) = mQ_{m-1}(0) = -\frac{m\pi^{3/2} \tan(\pi(m-1)/2)}{(m-1)\Gamma(\frac{2-m}{2})\Gamma(\frac{m-1}{2})}$ (see <http://functions.wolfram.com/07.07.20.0006.01>,

<http://functions.wolfram.com/07.07.03.0001.01>,

<http://functions.wolfram.com/07.10.20.0003.01>,

<http://functions.wolfram.com/07.10.03.0001.01>),

that m is an even integer ≥ 2 and that $\beta = 0$ so that f is up to a nonzero multiplicative constant the Legendre polynomial P_m . But, using [3, Corollary on p. 114] there is only one even integer ≥ 2 such that P_m has no critical point in $(0, 1)$, namely $m = 2$. We obtain $f(x) = c_2 P_2(x) = c_2 \frac{1}{2} (3x^2 - 1)$.

In order to obtain the claimed growth we calculate for $u_r(x) = u(rx)/r^\alpha$ and a.e. $r \in (0, \delta)$, using (3.14),

$$\begin{aligned} \left(\int_{\partial B_1^+(0)} \frac{1}{x_1} u_r^2 d\mathcal{H}^1 \right)' &= \frac{2}{r} \left(\int_{B_1^+(0)} \frac{1}{x_1} |\nabla u_r|^2 dx - \alpha \int_{\partial B_1^+(0)} \frac{1}{x_1} u_r^2 d\mathcal{H}^1 \right) \\ \begin{cases} \geq \frac{\kappa}{r} \int_{\partial B_1^+(0)} \frac{1}{x_1} u_r^2 d\mathcal{H}^1, & \alpha \in (0, 2), \\ \leq -\frac{\kappa}{r} \int_{\partial B_1^+(0)} \frac{1}{x_1} u_r^2 d\mathcal{H}^1, & \alpha > 2. \end{cases} \end{aligned}$$

Integrating we obtain the result. \square

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